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ABSTRACT

This is Part 1 of the first course in a series which focuses on building fundamental mathematical structures. The arithmetic studied in elementary school along with modular arithmetic is examined and set notation and mappings of sets are presented. Mathematical group structures are introduced. Points and numbers are related both on a line and in a lattice framework. (FL)

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Secondary School Mathematics
Curriculum Improvement Study

UNIFIED MODERN MATHEMATICS

COURSE I

PART I

SECOND PRINTING 1969

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Secondary School
Mathematics
Curriculum Improvement Study

UNIFIED MODERN MATHEMATICS
COURSE I
PART I

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Preface

Mathematics used to be considered as a study of separate branches called arithmetic, algebra, geometry, calculus and so on. Each of these subjects was studied separately for a year or more. Today mathematics is looked upon in a different manner--as a set of systems or structures which are common to all the classical branches. In these systems the ideas of set, operation, mapping, and relation are fundamental. In this manner of looking at mathematics, the subject gains a sense of unity, and the learning of it is made more efficient.

This is the first course of a series in which a start is made in building the fundamental structures. In this book we examine the arithmetic studied in the elementary school along with new clock arithmetics to see the nature of arithmetic and operational systems. To aid all subsequent learning we introduce the language of sets and mappings of sets. We develop the structure called a group, and an extended set of numbers called the integers which has an extended structure. We make mappings of points and relate points and numbers--on a line and in a lattice framework. Gradually arithmetic, algebra, and geometry merge into a unified study. As an additional study we consider probability and number theory.

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CHAPTER 1

FINITE NUMBER SYSTEMS

1.1 Jane Anderson's Arithmetic

Mr. Anderson was helping his daughter, who was in first grade, with her arithmetic homework. He asked, "Jane, what is seven plus three?"

Jane looked over her father's shoulder and soon answered, "Seven plus three is ten."

"That is correct," said her father. "Now, what is eleven plus two?"

Jane again glanced over her father's shoulder and said, "eleven plus two is one."

"My hearing must be bad," said her father. "I thought you said 'Eleven plus two is one.'"

"I did," said Jane.

Her father, of course, wanted to know why she made such a statement. Jane walked over to the clock on the shelf behind her father's shoulder. She explained how she found the sum of 7 and 3. She first pointed to "7" on the face of the clock and then moved her finger clockwise over three numerals. Since she was then pointing at "10" she said "Seven plus three is ten." Jane proceeded in the same way to find the sum of 11 and 2. She first pointed to "11" on the clock and then moved her finger clockwise over two numerals. Since she was then pointing at "1," she said, "Eleven plus two is one."

1.2 Clock Arithmetic

In answering questions relating to time we probably all have performed an operation quite similar to Jane's procedure. If you are asked what time it is three hours after seven o'clock, you naturally answer ten o'clock. We could express this result using the notation "$7 + 3 = 10$." But what if you were asked what time it is two hours after eleven o'clock? Now the answer is one o'clock, and using the same notation as above we have

$$11 + 2 = 1.$$

In the whole numbers it makes sense to assign 13 as the sum of 11 and 2, but on a clock it makes sense to assign 1 as this sum. To express the fact that nine hours after seven o'clock is four o'clock, we shall write

$$7 + 9 = 4.$$

Question: On a clock, what is expressed by "$11 + 6 = 5$"?

What would you reply if you were asked what time it is seven hours after eight o'clock? Is this the same as finding the sum of 8 and 7 using the arithmetic on a clock?

Question: What is the sum of 8 and 7 using the numbers on a clock? Explain how you obtained your answer.

One way to answer the above question would be to place a pointer on a clock face with the pointer directed at "12." In order to compute $8 + 7$, move the pointer clockwise through 8 intervals and then follow this by moving the pointer clockwise through 7 intervals. The pointer will then be directed at "3." Thus 3 is the number assigned as the sum of 8 and 7. (see Figure 1.1)

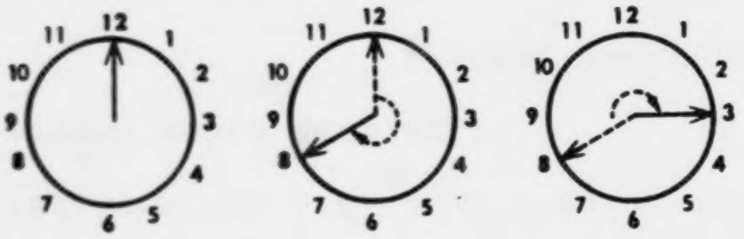


Figure 1.1: Using a dial to determine 8 + 7

The numbers represented on the face of a clock are the elements of the familiar set of clock numbers. We will represent the set {1,2,3,4,5,6,7,8,9,10,11,12} by the symbol "Z_{12}." The "Z" is suggested by the German word for number, Zahl. The subscript "$_{12}$" indicates the number of elements in this set.

1.3 Exercises

1. Compute the following sums by the procedure used on the clock:

(a) 9 + 4
(b) 7 + 9
(c) 7 + 8
(d) 5 + 6
(e) 10 + 11
(f) 11 + 10
(g) 1 + 12
(h) 12 + 1
(i) 11 + 11
(j) 12 + 9
(k) 9 + 12
(l) 12 + 12

2. Determine the clock numbers which, when placed in the boxes, yield true statements.

(a) 10 + □ = 6 (c) □ + 6 = 12

(b) 8 + □ = 3 (d) 11 + 12 = □

(e) □ + □ = □ (The same clock number is to be inserted in all three boxes.)

(f) □ + □ = 8 (The same clock number is to be inserted in both boxes. There are two answers.)

3. To avoid thinking of a moving pointer for each clock computation, we can construct a Z_{12} addition table similar to those you have seen for whole numbers. We indicate that the sum of 11 and 2 is 1 by placing "1" in the cell determined by row 11 and column 2. (see Figure 1.2) Examine the table and note how the sum of 7 and 3 is entered.

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	⑦	8	9	10	11	12	1
2												
3												
4												
5												
6												
7			10									
8												
9												
10												
11		1										
12												

Figure 1.2 Addition Table for Z_{12}

(a) Does the encircled "7" in the body of the table represent 1 + 6 or 6 + 1? Explain your answer.

(b) Discuss why the cells in row 1 were assigned the sums shown in Figure 1.2.

(c) Copy the table in Figure 1.2 and compute the entries for the second row, third row, etc. Do you notice any pattern emerging? Can you make any conjectures that can be tested?

(d) What interesting pattern relates the entries in the first column? The last column? The last row? How are these columns and rows related to other columns and rows?

(e) Complete the table.

(f) What differences can you see between the addition table constructed above and an addition table for the whole numbers?

1.4 $(Z_{12},+)$ and $(W,+)$

If we compare the set of clock numbers, Z_{12}, with the set of whole numbers

$$W = \{0,1,2,3,4,5,\ldots\}$$

we notice one immediate difference. The set of whole numbers is endless or infinite. If a set is not infinite, we say that it is finite.

It is important that you do not confuse a large finite set with an infinite set. Some common examples of finite sets are:

(a) the set of vowels in the English alphabet,

(b) the set of words in a dictionary,

(c) the set of all sentences which have ever been written.

Question 1: Give some examples of large finite sets.

Question 2: What is the largest finite set you can describe?

Question 3: What, besides W, would be an example of an infinite set?

The set of numbers that Jane Anderson was using when she said that 11 + 2 = 1 is a finite number system. Such finite systems have many interesting properties and applications. You have probably already observed similarities and contrasts between clock addition $(Z_{12},+)$ and whole number addition $(W,+)$. As we study the clock and other finite number systems, feel free to make guesses or conjectures about properties that appear familiar or unusual. You will find both if you are alert.

Using the familiar whole number addition, you will surely agree that the following computations are correct:

$10 + 7 = 17$ $\quad$ $7 + 10 = 17$

$3 + 6 = 9$ $\quad$ $6 + 3 = 9$

$11 + 4 = 15$ $\quad$ $4 + 11 = 15$

The pattern demonstrated by these six computations can be stated in general. For any whole numbers x and y,

$$x + y = y + x$$

This is the commutative property of addition in W.

Now use the table you constructed for Z_{12} addition to determine each of the following sums:

(a) 10 + 7 (d) 7 + 10

(b) 3 + 6 (e) 6 + 3

(c) 11 + 4 (f) 4 + 11

Does it appear that addition is commutative in Z_{12}? What patterns can you find in the $(Z_{12},+)$ table to support your answer?

The easiest whole number addition problems are those involving zero.

$9 + 0 = 9$ $0 + 9 = 9$

$756 + 0 = 756$ $0 + 756 = 756$

$27 + 0 = 27$ $0 + 27 = 27$

Because of the special way that zero behaves in whole number addition, it is called the additive identity element for W. For any whole number x,

$$x + 0 = 0 + x = x$$

Zero is not a clock number, but the Z does have an additive identity element. Look closely at the rows and columns of your $(Z_{12},+)$ table to find the clock number which acts the same in clock addition as zero in $(W,+)$.

1.5 Calendar Arithmetic

The traffic manager of the nation-wide Bee-Line Trucking Company of New York City was faced with the following situation. Trucks would return to New York City after extended road trips around the country and the manager had to arrange for garage space, the hiring of loaders and extra drivers, service on the truck engines, cargo assignment, etc. The manager found that he needed a fast way of determining the day of the week a truck would return if he knew (1) the day of the week that the truck left New York City and (2) the number of days that the truck would be on the road.

A typical problem was the following: A truck was to leave New York City for Indianapolis, Indiana (2 days); go on to Dallas, Texas (3 days); then to Washington, D.C. (4 days); and finally return to New York City (1 day). If this truck leaves New York City on Friday, on what day of the week does it return?

The manager soon struck on the idea of using a dial with days of the week assigned to numbers as in Fig. 1.3.

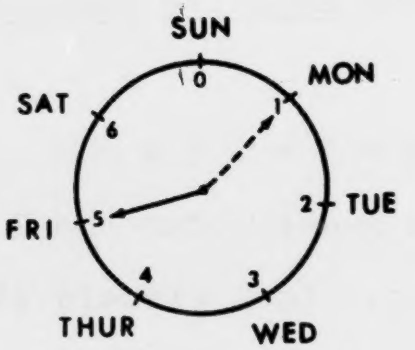


Figure 1.3: Calendar Numbers

To solve the above problem he proceeded as follows. Since the truck left on Friday, he set the pointer at "5." Since the total trip took 10 days (check this!) he then moved the pointer clockwise through ten intervals. The pointer then was directed at "1," so he concluded the truck would return on Monday.

The set of numbers used by the manager, {0,1,2,3,4,5,6} we will call the calendar numbers. We will refer to this set as Z_7.

Consider the following easy problem. If a truck left New York City on Thursday and returned six days later, then on what day of the week would it return? This problem can be interpreted as asking "What number in Z_7 should be assigned as the sum of 4 (the number associated with Thursday) and 6 (the time of the trip)?" We see that the sum obtained from use of the dial agrees with the obvious answer to the original problem, namely Wednesday. Thus in Z_7 we have that $4 + 6 = 3$.

1.6 Exercises

1. The manager of the Bee-Line Moving Company obtained the following data for one of his routes:

Depart	Arrive	Time of Travel (days)
New York City	Cleveland, Ohio	2
Cleveland	Jacksonville, Fla.	3
Jacksonville	Atlanta, Ga	1
Atlanta	El Paso, Texas	5
El Paso	Des Moines, Iowa	4
Des Moines	Chicago, Ill.	1
Chicago	New York City	3

Assume that a truck leaves New York City on a Wednesday.

(a) On what day of the week will it arrive in Jacksonville? In El Paso?

(b) On what day of the week will the truck return to New York City?

(c) If there is a two-day lay over in El Paso, on what day will it return to New York City?

(d) If a truck leaves on a Saturday, makes the complete route, lays over in New York City for two days, and then makes a second complete route, on what day of the week will it return to New York City?

2. Compute the following in Z_7

(a)	$6 + 1$	(f)	$5 + 4$	(k)	$0 + 6$
(b)	$2 + 6$	(g)	$5 + 5$	(l)	$1 + 6$
(c)	$3 + 5$	(h)	$5 + 6$	(m)	$2 + 5$
(d)	$4 + 2$	(i)	$6 + 6$	(n)	$5 + 2$
(e)	$4 + 5$	(j)	$0 + 1$	(o)	$0 + 0$

3. Determine the calendar numbers which, when placed in the boxes, yield true statements.

(a) $5 + \square = 3$ (c) $\square + 6 = 2$

(b) $2 + \square = 6$ (d) $\square + 3 = 1$

4. Copy the following addition table for Z_7 and complete it by filling in the remaining cells. Note how $4 + 6 = 3$ is recorded already.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1							
2							
3							
4							3
5							
6							

(a) Explain why we can dispense with the dial once the table is completed.

(b) Is there an identity element for $(Z_7,+)$? Explain your answer.

(c) Is addition commutative in $(Z_7,+)$?

(d) What interesting pattern relates the entries in row 1 and row 2? Row 2 and row 3?

(e) How are the entries in column 1 related to the entries in column 2?

(f) Explain why the upper right to lower left diagonal of the table turns out the way it does. Why is the upper left to lower right diagonal different?

5. Compare your tables for $(Z_{12},+)$ and $(Z_7,+)$

   (a) Do they have similar patterns relating row and column entries? If so, in what ways are they similar?

   (b) Do they have corresponding diagonal properties?

6. Compare $(Z_7,+)$ and $(W,+)$. How are they alike? How are they different?

7. It might be easier to compare $(Z_7,+)$ and $(Z_{12},+)$ if we replaced the additive identity, 12, in Z_{12} by 0. Then $Z_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$. Construct a new $(Z_{12},+)$ table as follows:

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1												
2												
3												
4												
5												
6												
7												
8								3				
9												
10												
11												

(a) Do the sums in the two $(Z_{12},+)$ tables agree -- with the exception that 0 takes the place of 12?

(b) Is $(Z_7,+)$ more similar to this new $(Z_{12},+)$ table than to the one in Section 1.3?

1.7 Open Sentences

How could you contrast the following mathematical sentences?

$$2 + 3 = 5 \qquad (1)$$

$$5 + 6 = 17 \qquad (2)$$

$$11 + 2 = \square \qquad (3)$$

It is obvious that sentence (1) is true in (W,+) and that sentence (2) is false in (W,+). However, we don't know whether (3) is true or false until the "□" is replaced by a symbol for a number. Both (1) and (2) are called mathematical statements since they are mathematical sentences that are either true or false (but not both).

Sentence (3) above, and others like it which contain a variable, appear frequently in mathematics. When we say that "□" is a variable in (3), we mean that the "□" can be replaced by a symbol for a number from a particular set of numbers. This set of numbers we call the domain of the variable.

If the domain of our variable is Z_{12}, then we could replace "□" in (3) by "1" and obtain a true sentence

$$11 + 2 = 1.$$

However, if the domain of our variable is W, then replacement of "□" by "1" would yield a false sentence. To obtain a true statement in W we should replace "□" by "13" since

$$11 + 2 = 13.$$

In dealing with sentences such as (3), always be aware of the domain of the variable(s) which you are considering.

A sentence such as "$11 + 2 = \square$," which is neither true nor false, is called an open sentence. Note that such a sentence will become either true or false after replacement of the box. It is easy to write down open sentences, that is, sentences which contain at least one variable and which are neither true nor false. Examples of open sentences are

$$\square + 2 = 6,$$
$$3 + 4 = \triangle,$$
$$7 + \square = 11,$$
$$\triangle + \square = 4,$$
$$\text{and } \square = 0.$$

An open sentence, like those above, with the equality sign is called an equation. Another kind of sentence used frequently in mathematics deals with the inequality relations "is less than" and "is greater than." For example, in the set of whole numbers we can write such sentences as "3 is less than 4" and "8 is greater than 6." We use the symbols "$<$" and "$>$" to denote, respectively, "is less than" and "is greater than." Thus we can rewrite the above sentences as "$3 < 4$" and "$8 > 6$." Examples of open sentences using these relations are

$$5 > \square + 1,$$
$$4 < \triangle + 6,$$
$$\text{and } \square > 0.$$

An open sentence with an inequality symbol, like those above, is called an inequation or an inequality.

Frequently you will be asked to solve an open sentence. This means that you are to determine those numbers in the domains of the variables which, when substituted for the variables, yield true statements. The set of numbers which yield true statements is called the solution set of the open sentence.

Question 1: Why is {0} the solution set of the open sentence □ + 4 = 4, where the domain of "□" is W?

Question 2: What is the solution set of the open sentence Δ + 3 = 1, if the domain of "Δ" is Z_{12}?

Question 3: What is the solution set for the open sentence 2 = □ + 5, where the domain of "□" is W?

You have probably already determined that the solution set for the open sentence in Question 3 has no members. It is an example of the empty or null set. Some other ways of describing the empty set are: the set of all men who are thirty feet tall, and the set of all whole numbers between $\frac{1}{2}$ and $\frac{3}{4}$. We usually indicate the empty set by the symbols "∅" or "{ }."

Question 4: Discuss why the solution set for Question 1, {0}, is not the same as the solution set for Question 3, ∅.

We have been using "□" and "Δ" to denote variables. It is more usual in mathematics to denote a variable by such a symbol as x, y, z, or n. If we use these symbols to rewrite the examples of open sentences given earlier, they would be

$$x + 2 = 6,$$

$$3 + 4 = y,$$

$$x + y = 4,$$

and $5 > n + 1$.

Let us review some of the above ideas by considering the following.

Example 1. Let the domain of the variable be W. If we are asked to solve $\square + 5 = 12$ and $\square$ is replaced by 7, we obtain $7 + 5 = 12$, which is true. Hence 7 is the solution of the open sentence, or $\{7\}$ is the solution set, since no other replacement from W would give a true statement.

Example 2. Let the domain of the variable be Z_7. In the open sentence $\square + 4 = 3$, if $\square$ is replaced by 6, we obtain $6 + 4 = 3$ which is true. Hence 6 is the solution or $\{6\}$ is the solution set, since no other replacement of $\square$ makes the sentence true.

Example 3. Let the domain of the variable be W. For the open sentence $\square + 4 = 3$, we find that there is no replacement from W which yields a true sentence. Hence there is no solution in W or the solution set is the empty set.

Example 4. Let the domain of the variable be Z_{12} with "12" replaced by "0." That is $Z_{12}=\{0,1,2,3,4,5,6,7,8,9,10,11\}$. In the equation $x + x = 10$, if x is replaced by 5 we obtain $5 + 5 = 10$, which is true. Hence 5 is a solution of the open sentence. If x is replaced by 11, we obtain $11 + 11 = 10$, which is also true. Hence 11 is another solution of the open sentence. If we replace x by any other clock number, we will obtain false statements. We conclude that the solution set of the given equation is $\{5,11\}$.

Example 5. If the domain of the variable is W, then the solution set of the inequation $n + 3 < 8$ is $\{0,1,2,3,4\}$.

Example 6. If the domain of the variable is W, then the solution set of $y + 3 > 8$ is the set of whole numbers greater than 5, that is, $\{6,7,8,9,\ldots\}$

1.8 Exercises

1. Label each of the following sentences as true, false, or open in (W,+):

   (a) $12 + 3 = 14 + 1$

   (b) $395 + 682 > 1051 + 86$

   (c) $\square + 87 = 91$

   (d) $765 = 700 + 65$

   (e) $\square + 0 = \square$

   (f) $\square = 1 + \square$

   (g) $1262 + 2384 = 2126 + 3248$

   (h) $\Delta + \square = \square + \Delta$

2. Explain why or why not the following are true sentences:

   (a) $11 + 7 = 5$ in $(Z_{12},+)$

   (b) $5 + 5 = 3$ in $(Z_7,+)$

   (c) $2 + 3 = 4 + 1$ in $(Z_7,+)$

   (d) $10 + 8 = 2 + 4$ in $(Z_{12},+)$

3. Solve the following open sentences given that the domain of the variable is W: (If the same symbol for a variable appears more than once in a sentence, then it represents the same number each time that it occurs.)

   (a) $\square + 7 = 15$ (f) $z + 1 > 3$

   (b) $\square + \square = 6$ (g) $\square + 1 > 1$

   (c) $13 = \square + 1$ (h) $n < 4$

   (d) $7 + x = 15$ (i) $x + 1 < 11$

   (e) $35 + y = 25$ (j) $x + x = 10$

4. Solve the following open sentences given that the domain of the variable is Z_7:

   (a) $2 + 5 = \square$ (d) $6 + x = 1$

   (b) $\square + 3 = 3$ (e) $6 + 6 = y$

   (c) $\square + 4 = 1$ (f) $\square + \square = 5$

5. Solve the following open sentences given that the domain of the variable is Z_{12}:

   (a) $5 + 8 = \square$ (d) $9 + z = 2$

   (b) $4 + 3 = x$ (e) $7 + x = 7$

   (c) $y + 3 = 9$ (f) $x = x + 1$

6. Using the symbol "x," write an open sentence whose solution set is $\{6\}$ where the domain of the variable is

   (a) W (b) Z_7 (c) Z_{12}

7. Using the symbol "x," with the domain W, write an open sentence whose solution set is

   (a) W (b) the empty set (c) the set {0}.

8. If two open sentences have the same solution set then we say they are equivalent. Find the solution set of each pair of sentences below. Then determine if the sentences are equivalent. Let the domain of the variables be W.

   (a) $12 + \square = 31$
   $\square + 7 = 36$

   (b) $x + 3 = 3$
   $4 + x = 2$

   (c) $z + 2 < 5$
   $8 > 2 + 5$

   (d) $x = 3 + 0$
   $0 + 3 = x$

   (e) $y = 2 + 0$
   $y + 2 = 0$

1.9 New Clocks

In Section 1.2 we constructed a "clock arithmetic" for the familiar clock which uses twelve numerals on its face. If we wanted a clock with exactly seven numerals on its face, one possibility would be to use the system constructed in Section 1.5. There, for the calendar numbers, we made use of a pointer and the set {0,1,2,3,4,5,6}. We have already explored how the arithmetic for such a 7-clock would proceed. (See Section 1.6 Exercises 1,2,3, and 4.)

Our previous work with a 12-clock and a 7-clock indicates that we could easily construct other clocks. For example, a 4-clock is suggested by a lamp switch which has four positions: "Off," "Low," "Medium," and "Bright." If we assign, in order, the numerals "0," "1," "2," and "3" to these four positions and again introduce a pointer we can draw a picture of such a 4-clock. (We will refer to the set $\{0,1,2,3\}$ as Z_4.)

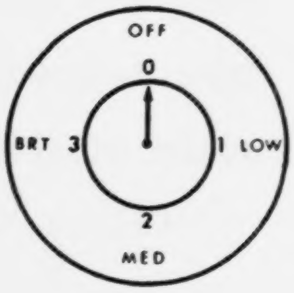


Figure 1.4: A 4-clock

1.10 Exercises

1. (a) Construct an addition table for the 4-clock.
   (b) Use your addition table to do these computations.
   (1) $1 + 2$ (3) $2 + 1$ (5) $2 + 2$ (7) $1 + 3$
   (2) $3 + 1$ (4) $3 + 3$ (6) $0 + 3$ (8) $3 + 2$
   (c) Use your table to determine the position a lamp switch would be in if, starting from the "Off" position we turned it clockwise through 3 intervals and followed this with another clockwise turn through 2 intervals.
   (d) How can an examination of the table help you decide whether or not addition in Z_4 is commutative?
2. (a) Compare $(Z_4,+)$ with $(Z_7,+)$.
   (b) Compare $(Z_4,+)$ with $(W,+)$.

3. (a) Make up an addition table for a 5-clock using the set $Z_5 = \{0,1,2,3,4\}$.

   (b) Using your addition table for Z_5 compute the following:

   (1) $2 + 4$
   (2) $1 + 4$
   (3) $3 + 2$
   (4) $4 + 4$
   (5) $3 + 3$
   (6) $4 + 3$

   (c) Compare $(Z_5,+)$ with $(Z_4,+)$.

   (d) A burner on an electric stove is controlled by a circular switch. The five possible positions are arranged and labeled in the following order: "Off," "Simmer," "Low," "Medium," and "High." Make up three problems which the table constructed in 3 (a) can help you solve (see Exercise (b) above).

4. (a) What kind of a clock is suggested by the channel selector knob on a T.V. set?

   (b) Can you find any everyday applications of any clocks that have been, or could be, constructed?

5. Examine the tables for addition of clock numbers that you have constructed.

   (a) What properties of addition tables can you find which make them easy to construct?

   (b) Make up an addition table for $Z_6 = \{0,1,2,3,4,5\}$.

   NOTE: Keep this table for $(Z_6,+)$, and all other tables that you construct, for future use.

1.11 Rotations

In Figure 1.5 a six sided geometric figure called a regular hexagon is drawn in a circle with center point labeled C. We say that C is also the center of the regular hexagon. The points of the hexagon which are on the circle are called the vertices of the regular hexagon and are labeled 0,1,2,3,4 and 5.

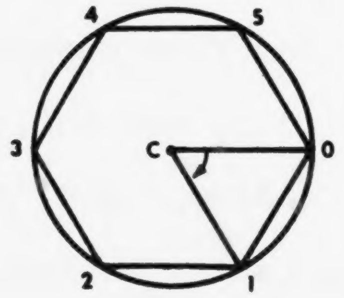


Figure 1.5: A regular hexagon inscribed in a circle

If we keep the center C of the hexagon fixed and rotate the hexagon in a clockwise direction until the vertex at "0" is moved to "1," and the vertex at "1" is moved to "2," etc. then we say that we have a rotation of the regular hexagon through 60° about C. Let us denote this rotation by r_1 ("r" to suggest rotation and "1" to suggest 1 interval of 60°). The point C is called the center of rotation. If our rotation about C passes through 120° (or 2 intervals) we shall denote this by r_2. Another way to view r_2 is as the result of performing an r_1 rotation and then following it with another r_1 rotation. What would r_3 denote? What would r_4 denote? r_6 is of particular interest since then we would be in what position?

Let us examine what r_5 denotes. We could say that in r_5 we have an instruction to rotate the regular hexagon about C through five intervals or 300°. If we followed the r_5 instruction by the r_1 instruction, then we would have completed a rotation of 360° which is, of course, one complete rotation. If we examine the subscripts of r_5 and r_1 we find that r_6 is suggested as the result of following r_5 by r_1. But r_6, considered alone, means that we are once again at our original position. In other words, the instructions r_6 and r_0 have the same effect. We choose to call such an instruction "r_0." We say that r_0 is the result of following r_5 by r_1 or that r_0 is assigned to r_5 and r_1.

1.12 Exercises

1. What result would you assign to the following?
   (a) r_1 followed by r_2
   (b) r_1 followed by r_3
   (c) r_2 followed by r_5
   (d) r_4 followed by r_2
   (e) r_3 followed by r_0
2. (a) Why was r_0 said to be the same instruction as r_6?
   (b) What is the result if any instruction is followed by r_0?
   (c) What special name might be given to r_0? Justify your answer.

3. Examine the partially completed table below.

rot.	r_0	r_1	r_2	r_3	r_4	r_5
r_0						r_5
r_1		r_2				
r_2					r_0	
r_3						
r_4						
r_5			r_1			

You will note that "r_2," "r_0," "r_1," and "r_5" have been made entries in the table.

(a) Explain the following:

(1) Why was r_2 assigned to r_1 and r_1?

(2) Why was r_0 assigned to r_2 and r_4?

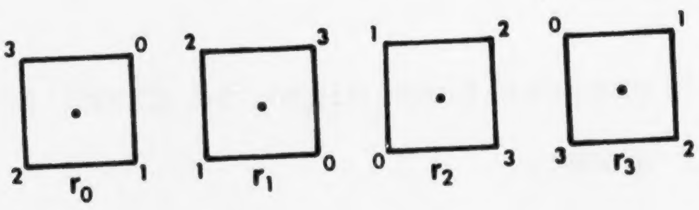
(3) Why was r_1 assigned to r_5 and r_2?

(4) Why was r_5 assigned to r_0 and r_5?

(b) Complete the table.

(c) In what way, if any, does the above table suggest a clock arithmetic?

4. Let the rotations of a square through 90°, 180°, 270° and 360° about a fixed center be designated by "r_1," "r_2," "r_3," and "r_0."



(a) If we label the vertices of the square with the numerals "0," "1," "2," and "3" as in r_o, then describe what is indicated by r_1, r_2, and r_3 (Figure 1.7).

(b)

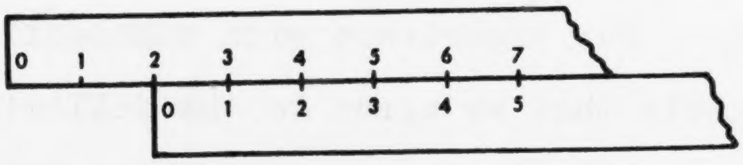
rot.	r_o	r_1	r_2	r_3
r_o				
r_1				r_o
r_2				
r_3				

In the above table why has r_o been assigned to r_1 and r_3? Copy this table and complete it (as was done in Exercise 3 (b) above).

(c) In what ways does this system of rotations correspond to a system studied earlier?

*5. A regular heptagon has 7 sides. Explain how we could use a similar system studied earlier to help complete a table for the rotations of this figure.

*6. Place two rulers next to each other and determine how a slide-rule can be developed that will let you add certain whole numbers.



How can the above be used to find the sum of 2 and 5?

For finite arithmetics the most appropriate slide-rules are circular. Why? Using two concentric cardboard disks (that is, circles and their interiors) see if you can devise a "slide rule" that will enable you to add in Z_{12}

(or any other clock arithmetic).

1.13 Subtraction in Clock Arithmetic

From your experience in arithmetic you know that given any two whole numbers, x and y it is always possible to find their sum, x + y. For instance,

$$75 + 36 = 111$$

$$235 + 831 = 1066$$

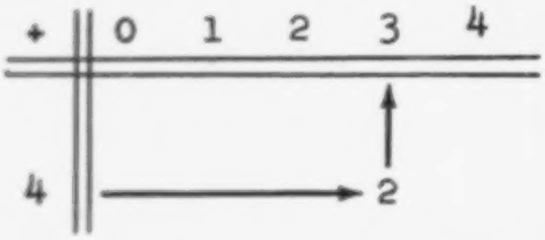
$$1,637,428 + 78,423 = 1,715,851$$

Furthermore, the sum is always unique. There is only one number which is assigned as the sum of x and y. Subtraction, on the other hand, is a restricted operation. For example, "2 - 4" does not name an element in the set W since there is no whole number which we can add to 4 to yield 2.

In each of the clock systems studied so far, addition has always been a possible operation. Is subtraction also always possible in the clock systems, or must it be restricted as it was in W?

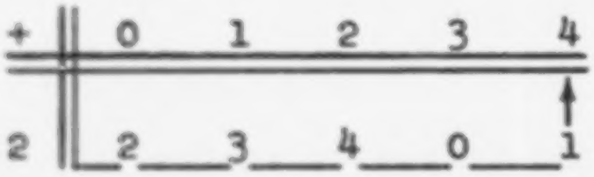
In particular, let us determine if "2 - 4" names a number in the set Z_6. Our experience with subtraction in the whole numbers suggests that we agree to the following:
If there is one and only one number in Z_6 which when added to 4 yields 2 we shall say that "2 - 4" names this number. In order to see that such a number does exist, we can make use of the addition table for $(Z_6,+)$. First locate "4" at the left in the table for $(Z_6,+)$. (See the partially completed table below.)

Move across the row headed by this "4" until you find the entry "2" in the body of the table. The number heading the column in which we find this "2" is 3.



We conclude that $2 - 4 = 3$ in Z_5 because 3 is the number in Z_5 which when added to 4 yields 2.

In order to find the number named by "1 - 2" in Z_5 we seek the number in Z_5 which when added to 2 yields 1. From the table for $(Z_5,+)$ we see that $2 + 4 = 1$.



Thus we conclude that $1 - 2 = 4$ in Z_5. Note that 4 is the only number in $(Z_5,+)$ which when added to 2 yields 1. Thus the difference 4 is unique. There is one and only one number, 4, which when added to 2 yields 1.

In order to find the number named by "4 - 1" in Z_5 we seek the one and only one number which when added to 1 yields 4. Since $1 + 3 = 4$ in $(Z_5,+)$ we conclude that $4 - 1 = 3$ in Z_5. Do you see that the difference 3 is unique?

In order to compute $0 - 3$ in Z_5 we seek the one and only one number which when added to 3 yields 0 in $(Z_5,+)$. Since $3 + 2 = 0$ in $(Z_5,+)$ we conclude that $0 - 3 = 2$ in Z_5.

NOTE: In many exercises in this chapter we shall use the word "compute" to mean "find the simplest name." For example, we expressed the result of computing 0 - 3 in Z_5 by using the simplest name "2;" that is, 0 - 3 = 2 in Z_5.

1.14 Exercises

1. Using your addition table for $(Z_5,+)$ find the simplest name for

(a) 1 - 4	(f) 3 - 0
(b) 4 - 3	(g) 0 - 4
(c) 3 - 4	(h) 2 - 3
(d) 4 - 2	(i) 1 - 3
(e) 4 - 4	(j) 0 + 2

2. Below is a partially completed subtraction table.

-	0	1	2	3	4
0				2	
1			④		
2					
3					
4		3			

Note that the encircled 4 in the table indicates that 1 - 2 = 4 in Z_5.

(a) Copy the above table and compute the remaining entries in this subtraction table.

(b) Is subtraction always possible in the set Z_5? Explain your answer.

(c) How does subtraction in Z_5 compare with subtraction in W?

3. You recall that we introduced addition for our finite sets by making use of a clock. Then we constructed addition tables. However, subtraction in Z_5 was first introduced by using the idea of a table. The following exercises relate this subtraction to a 5-clock.

   (a) If the pointer of a 5-clock is placed on the numeral 2 and then moved counter-clockwise through 3 intervals, then at what numeral is the pointer directed? What subtraction problem in Z_5 does this solve?

   (b) Can you state how we could find the simplest name for "1 - 2" on a 5-clock?

   (c) Find the simplest name for each of the following and then describe how to check your results using a 5-clock.

      (1) 1 - 4
      (2) 0 - 3
      (3) 4 - 1
      (4) 2 - 4

4. Use your table for $(Z_5,+)$ to find the simplest name for

   (a) 5 - 2
   (b) 2 - 5
   (c) 4 - 1
   (d) 1 - 4
   (e) 3 - 4
   (f) 0 - 4
   (g) 2 - 3
   (h) 1 - 5

5. Use your table for $(Z_7,+)$ to find the simplest name for

   (a) 1 - 6
   (b) 5 - 6
   (c) 4 - 6
   (d) 2 - 6
   (e) 6 - 2
   (f) 6 - 0
   (g) 0 - 6
   (h) 6 - 6
   (i) 3 - 5
   (j) 3 - 6
   (k) 0 - 0
   (l) 1 - 4

6. Is there an identity element for subtraction in

   (a) Z_5?

   (b) Z_6?

   (c) Z_7?

   Justify your answer.

7. Is subtraction commutative in

   (a) Z_5?

   (b) Z_6?

   (c) Z_7?

   Justify your answers.

8. Solve the following open sentences given that the domain of the variable is Z_5:

   (a) $2 - 4 = x$

   (b) $y - 4 = 1$

   (c) $3 - z = 1$

   (d) $3 - x = 2$

   (e) $1 - 4 = y$

   (f) $2 - 3 = z$

   (g) $3 - x = 3$

   (h) $3 - y = 4$

   (i) $3 - z = 0$

   (j) $0 - x = 0$

   (k) $1 - 3 = y$

   (l) $z - 4 = 4$

9. Solve the following open sentences where the domain of the variable is Z_6:

   (a) $3 - 5 = x$

   (b) $2 - 5 = y$

   (c) $1 - 2 = z$

   (d) $0 - x = 2$

   (e) $5 - 2 = y$

   (f) $y - 4 = 4$

   (g) $z - 4 = 5$

   (h) $1 - x = 3$

   (i) $0 - y = 2$

   (j) $0 - z = 0$

1.15 Multiplication in Clock Arithmetic

We shall now consider how multiplication will be defined in clock arithmetic. From your previous study of the whole numbers, you know that given the pair of whole numbers 3 and 4 you would assign the whole number 12 to this pair as their product. In short, $3 \cdot 4 = 12$ in $(W, \cdot)$.

But how should we define the product $3 \cdot 4$, for example, in Z_5? Even though we could assign any number in Z_5 as this product, let us agree that the product $3 \cdot 4 = 2$ in Z_5. Why we select 2 as the product can be seen if we note the following relationship between $(W, \cdot)$ and the 5-clock. In $(W, \cdot)$ we have that $3 \cdot 4 = 12$. If we place the pointer of a 5-clock on "0" and then move it clockwise through 12 intervals the pointer will be directed at "2." Using this result we define $3 \cdot 4 = 2$ in Z_5.

We shall use the above relationship between $(W, \cdot)$ and the 5-clock to define $2 \cdot 4$ in Z_5. Since $2 \cdot 4 = 8$ in $(W, \cdot)$ we move the pointer of a 5-clock clockwise through 8 intervals from "0." The pointer is then directed at "3." Using this result we define $2 \cdot 4 = 3$ in Z_5.

How should we define $4 \cdot 4$ in Z_5? Since $4 \cdot 4 = 16$ in $(W, \cdot)$ we move the pointer of a 5-clock clockwise through 16 intervals from "0." The pointer is then directed at "1." Thus we define $4 \cdot 4 = 1$ in Z_5.

There is another approach to multiplication in Z_5 that does not use the idea of a clock. The key idea in this second approach is that of remainder. Do you remember how this term was used in your earlier study of mathematics? For example, if the whole number 8 is divided by the whole number 5 we obtain a quotient of 1 and a remainder of 3. Recall that in defining $2 \cdot 4$ in Z_5 we moved a pointer on a 5-clock through 8 intervals and the pointer was directed at "3." We see that in this example the resulting product given by use of the clock is precisely the remainder obtained when 8 is divided by 5. Will the remainder approach continue to give results equivalent to the clock approach?

We can test to see if the products defined earlier on pairs of numbers in Z_5 are related to "remainders." For example, earlier we defined $4 \cdot 4$ to be 1 in Z_5. By the remainder approach we first note $4 \cdot 4 = 16$ in $(W, \cdot)$. Then we divide 16 by 5 and obtain a quotient of 3 and a remainder of 1. If we disregard the quotient and examine the remainder we see that this remainder, 1, is the same number which we defined earlier as the product of 4 and 4 in Z_5.

If we apply this remainder approach in order to define $3 \cdot 4$ in Z_5, we proceed as follows: Compute $3 \cdot 4$ in $(W, \cdot)$ and obtain 12; divide 12 by 5 obtaining a quotient of 2 and a remainder of 2: we record the remainder 2 as the product: thus $3 \cdot 4 = 2$ in Z_5. Again this agrees with an earlier result.

Let us now indicate a scheme whereby we can assign a "product" to any pair of numbers in Z_5. If $\underline{a}$ and $\underline{b}$ are any two numbers in $Z_5 = \{0,1,2,3,4\}$, we first form their product in $(W, \cdot)$. This product is then divided by 5, and we note the remainder. From above, we know that this remainder is also a number in Z_5. We record this remainder and call it "the product of a and b," which we write as "$a \cdot b$."

Example 1. If we wish to compute the product of 3 and 3 in Z_5 we note first that $3 \cdot 3 = 9$ in $(W, \cdot)$. When 9 is divided by 5 we obtain a quotient of 1 and a remainder of 4. We disregard the quotient and record the remainder 4 as the product we are seeking.

Thus,

$$3 \cdot 3 = 4 \text{ in } (Z_5, \cdot)$$

Example 2. The product of 2 and 2 in $(Z_5, \cdot)$ is found by noting that $2 \cdot 2 = 4$ in $(W, \cdot)$ and 4 divided by 5 yields a quotient of 0 and a remainder of 4. We disregard the quotient and record the remainder 4 as the product.

Thus,

$$2 \cdot 2 = 4 \text{ in } (Z_5, \cdot).$$

Example 3. The product of 3 and 0 is 0 in Z_5 because $3 \cdot 0 = 0$ in $(W, \cdot)$ and 0 divided by 5 yields a remainder of 0.

1.16 Exercises

1. Below is a partially completed multiplication table for pairs of numbers in Z_5. Some of the products obtained above have been recorded.

$\cdot$	0	1	2	3	4
0					
1					
2			4		3
3	0			4	2
4					1

(a) Copy the above table and compute the remaining entries.

(b) Do you notice any interesting patterns relating the entries of a single row or column? Relating pairs of rows or columns?

2. Solve the following open sentences in $(Z_5, \cdot)$:

(a) $3 \cdot x = 1$

(b) $4 \cdot x = 4$

(c) $y \cdot 2 = 0$

(d) $0 \cdot x = 0$

(e) $3 \cdot x = 2$

(f) $3 \cdot x = 0$

(g) $4 \cdot x = 2$

(h) $1 \cdot y = 3$

(i) $3 \cdot x = 3$

(j) $4 \cdot z = 1$

(k) $4 \cdot y = 3$

(l) $0 \cdot x = 2$

3. If we wish to construct a multiplication table for $(Z_4, \cdot)$ we record remainders resulting from division by 4. Thus, to determine the product of 2 and 3 in $(Z_4, \cdot)$ we note that 6, the product of 2 and 3 in $(W, \cdot)$, when divided by 4, gives quotient 1 and remainder 2.

Thus,

$$2 \cdot 3 = 2 \text{ in } (Z_4, \cdot)$$

(a) Copy and complete the following (multiplication) table for $(Z_4, \cdot)$

$\cdot$	0	1	2	3
0				
1				
2				2
3				

(b) Do you notice any interesting patterns relating the entries of a single row or column? Relating pairs of rows and columns?

4. Construct multiplication tables for $(Z_8, \cdot)$ and for $(Z_7, \cdot)$.

   (a) Examine the tables for $(Z_4, \cdot)$ and $(Z_8, \cdot)$. In what ways are these tables similar? In what ways are they different?

   (b) Examine the tables for $(Z_8, \cdot)$ and $(Z_7, \cdot)$. What properties do they have in common? Can you find some essential differences between the tables?

5. Solve the following open sentences

   (a) In $(Z_4, \cdot)$:

   (1) $3 \cdot x = 2$
   (2) $2 \cdot y = 2$
   (3) $2 \cdot z = 0$
   (4) $3 \cdot x = 1$
   (5) $0 \cdot x = 0$
   (6) $0 \cdot z = 3$
   (7) $1 \cdot y = 3$
   (8) $2 \cdot x = 3$

(b) In $(Z_7, \cdot)$:

(1) $6 \cdot x = 3$ (2) $3 \cdot y = 5$
(3) $5 \cdot y = 1$ (4) $y \cdot 2 = 0$
(5) $4 \cdot y = 2$ (6) $x \cdot 0 = 5$
(7) $6 \cdot z = 5$ (8) $6 \cdot x = 0$

(c) In $(Z_6, \cdot)$:

(1) $5 \cdot x = 3$ (2) $4 \cdot y = 5$
(3) $4 \cdot z = 3$ (4) $4 \cdot x = 0$
(5) $4 \cdot h = 2$ (6) $4 \cdot z = 1$
(7) $3 \cdot x = 3$ (8) $3 \cdot y = 0$
(9) $3 \cdot y = 5$ (10) $2 \cdot z = 2$
(11) $2 \cdot x = 0$ (12) $2 \cdot y = 4$

1.17 Comparison of $(W, \cdot)$ and Clock Multiplication

In Section 1.4 we found that addition of whole numbers was similar in many respects to addition of clock numbers. Each operation was commutative and each had an identity element. From your earlier work in arithmetic, you probably recall similar properties of whole number multiplication:

For any whole numbers x and y, $x \cdot y = y \cdot x$.

This is the commutative property of multiplication in W.

For every whole number z, $1 \cdot z = z \cdot 1 = z$.

We therefore call 1 the multiplicative identity in W.

Examine the second row and second column of your $(Z_6, \cdot)$ table. What is true when a number in Z_6 is multiplied by 1? What is true when 1 is multiplied by any number in Z_6?

How does examination of the table provide evidence of a commutative property for $(Z_6, \cdot)$? Try to formulate a reason why you think commutativity is or is not a property of $(Z_5, \cdot)$. Then repeat your investigation in the $(Z_4, \cdot)$ table. Does it have an identity and commutativity? What about $(Z_5, \cdot)$, $(Z_7, \cdot)$?

If your multiplication tables were carefully constructed, you will be led to observe that multiplication is commutative in $(Z_5, \cdot)$, and indeed in all the clock "arithmetics," and that in each, 1 is the multiplicative identity.

Multiplication of whole numbers has another interesting property which has an analogy in clock arithmetic. Examine the entries in the first row and column of the $(Z_5, \cdot)$, $(Z_4, \cdot)$, $(Z_6, \cdot)$ and $(Z_7, \cdot)$ tables. What is true when any clock number is multiplied by 0? What is true when 0 is multiplied by any clock number? Again, if your multiplication tables were constructed carefully, you will have observed that for any clock number $\underline{x}$,

$$0 \cdot x = x \cdot 0 = 0.$$

This property is the multiplication property of zero, and it is also true in $(W, \cdot)$ if $\underline{x}$ is any whole number.

We have seen three multiplication properties which hold in each clock arithmetic and the arithmetic of whole numbers:

(1) Multiplication has the commutative property.

(2) There is a multiplicative identity, denoted by "1" in each arithmetic.

(3) The multiplication property of zero holds.

The tables for $(Z_4, \cdot)$ and $(Z_6, \cdot)$ differ in at least one essential way from the tables for $(Z_5, \cdot)$ and $(Z_7, \cdot)$. If we disregard the row and column headed by "0" in $(Z_5, \cdot)$ or in $(Z_7, \cdot)$ we see that there is no repetition of the entries in the remaining rows and columns. But there is a repetition of some entries in the rows and columns of the tables for $(Z_4, \cdot)$ and $(Z_6, \cdot)$. An interesting problem is to predict what other clock multiplication systems will be of the "$(Z_6, \cdot)$ type" and which of the "$(Z_7, \cdot)$ type." What other tables besides those for $(Z_4, \cdot)$ and $(Z_6, \cdot)$ have the "repetition of entries" property? Experiment by examining the tables for $(Z_3, \cdot)$ and $(Z_8, \cdot)$.

As you examine these tables, consider the following questions:

(a) Does the $(Z_3, \cdot)$ table behave as the "$(Z_6, \cdot)$ type" table or as the "$(Z_7, \cdot)$ type" table with regard to entries?

(b) Does the $(Z_8, \cdot)$ table behave as the "$(Z_6, \cdot)$ type" table or as the "$(Z_7, \cdot)$ type" table?

(c) Can you detect any pattern developing? Can you make a conjecture concerning which tables behave as did the $(Z_6, \cdot)$ table and which behave as did the $(Z_7, \cdot)$ table?

(d) After you have a conjecture, test it out by considering the multiplication table for $(Z_9, \cdot)$. Does your conjecture still hold true?

(e) You might want to experiment further in order to find a pattern that predicts how clock multiplication tables will behave as regards "repetition of the entries." Can you predict which elements in a given clock number system will repeat as entries in the multiplication table?

If you have become very familiar with the methods of clock arithmetic, try the following research problem involving secret codes:

Suppose you wanted to make up a code in order to send a secret message to a friend. One type of code is called a substitution code. In such a code one letter of the alphabet is substituted for another letter by means of a key or by writing some formula which indicates how the substitutions are made. For example, if each letter is replaced by the one that follows it in the alphabet, then we can describe this substitution by the formula

$$x' = x + 1 \text{ in } (Z_{26}, +).$$

This means that any letter x is replaced by the letter x' (read "x prime") which follows it. Thus "b," "k," and "q" would be replaced, respectively, by "c," "l," and "r."

(a) How would you encode, that is, put into code, the word "DANGER ?"

(b) How would you decode the word "IFMQT"

(c) By what letter would you replace "z? "

(d) Why was Z_{26} used in the above formula?

(e) What would the formula be if "a" is replaced by "d," "b" by "e," etc.?

(f) A similar type of code in which we use only number symbols, and which we have special symbols for "space," "comma," and "period" is the following: Assign 0 to a, 1 to b, 2 to c,..., 23 to x, 24 to y, 25 to z, 26 to "space," 27 to "comma," and 28 to "period." Using the formula $x' = x + 1$ in $(Z_{29}, +)$ we would then encode "JAMES BOND" as
10 1 13 5 19 27 2 15 14 4.
(Note that 27 represents the word "space.")
Examine the following coded messages. The system $(Z_{29}, +)$ is used. Can you find the formula that tells how substitutions are made and then decode the message? Hint: Try $x: = x + n$, $n = 0, 1, 2$, etc.

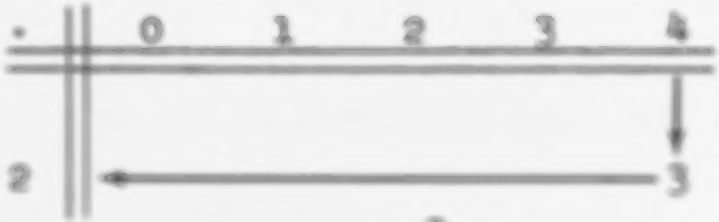
(1) 6 14 14 6 14 11 26 8 18 26
19 7 4 26 18 15 24 28

(2) 11 16 0 7 16 9 14 11 21 10 0
22 7 26 22 1 0 22 10 7 0
14 7 22 22 7 20 0 7 0
17 5 5 23 20 21 0
15 17 21 22 0
17 8 22 7 16 2

(g) Explain why using a formula such as $x' = 2 \cdot x$ in $(Z_{26}, \cdot)$ will not work. What goes wrong? Would the same problem arise if we had used the same formula $x' = 2 \cdot x$ but instead worked in $(Z_{29}, \cdot)$?

1.18 Division in Clock Arithmetic

Suppose we wish to divide 3 by 4 in Z_5. Let us recall our experience with division in the whole numbers and use the symbols "3 ÷ 4" or " $\frac{3}{4}$ " to denote such a quotient. We read these symbols as "3 divided by 4" or "3 over 4" and assume they mean the same thing. What follows is suggested by how the division process was carried out in W. Let us agree that to evaluate $\frac{3}{4}$ in Z_5 we seek the one and only one number in Z_5 which when multiplied by 4 yields 3. The partially completed multiplication table given below indicates how we can search for such a number. First locate "4" at the top of the table. Move down the column headed by this "4" until you find the entry "3." The number heading the row in which we find this "3" is 2. Thus we conclude that " $\frac{3}{4}$ " and "2" are two names for the same number in $(Z_5, \cdot)$.



We can express this by the equation $\frac{3}{4} = 2$. Note that there is one and only one number in Z_5, namely 2, which is equal to $\frac{3}{4}$, since "3" appears once and only once in the column headed by "4."

Example 1. How do we evaluate $\frac{4}{3}$ in Z_5? In order to evaluate $\frac{4}{3}$ in Z_5 we proceed as follows. We ask, "Does there exist one and only one number in Z_5 which when multiplied by 3 in $(Z_5, \cdot)$ yields 4?"

$3 \cdot 3 = 4$ in $(Z_5, \cdot)$, and no other number in Z_5 has this property. Thus we conclude that $\frac{4}{3} = 3$ in Z_5.

Example 2. How do we evaluate $\frac{0}{3}$ in Z_5? We seek the one and only one number in Z_5 which when multiplied by 3 in $(Z_5, \cdot)$ yields 0. Since $0 \cdot 3 = 0$ in $(Z_5, \cdot)$, and no other number in Z_5 has this property, we conclude that $\frac{0}{3} = 0$ in Z_5.

1.19 Exercises

1. Find the simplest names in Z_5 for

(a) $\frac{3}{2}$	(f) $\frac{2}{3}$	(k) $\frac{1}{2}$
(b) $4 \div 2$	(g) $2 \div 4$	(l) $2 \div 1$
(c) $\frac{4}{1}$	(h) $\frac{1}{4}$	(m) $\frac{1}{3}$
(d) $1 \div 3$	(i) $\frac{0}{1}$	(n) $\frac{3}{1}$
(e) $\frac{4}{4}$	(j) $\frac{0}{4}$	(o) $\frac{0}{0}$

2. Try to find the simplest names in Z_5 for

(a) $\frac{1}{0}$	(c) $\frac{3}{0}$
(b) $\frac{2}{0}$	(d) $\frac{4}{0}$

3. Is division an unrestricted operation in Z_5? That is, is it always possible to find the quotient of two given Z_5 numbers? How is division like or unlike subtraction in this respect?

4. Solve the following open sentences in Z_5:

   (a) $\frac{1}{2} \cdot x = 4$ (c) $4 + y = 1$

   (b) $\frac{3}{4} = 2 \cdot x$ (d) $\frac{x}{3} = \frac{3}{2}$

5. What do you notice when you try to construct a division table for pairs of numbers in Z_5? How does this table compare with the multiplication table for $(Z_5, \cdot)$?

6. In Z_6 we have $\frac{4}{2} = 5$ because $5 \cdot 2 = 4$ in $(Z_6, \cdot)$. Make an investigation of the process of division in other pairs of numbers in Z_6. Write a brief report on your findings.

1.20 Inverses in Clock Arithmetic

If we examine the tables given below for $(Z_4, +)$ and $(Z_4, \cdot)$ certain properties are easily found.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

For example, there is an additive identity element, namely 0, in $(Z_4, +)$. There is also a multiplicative identity element, namely 1, in $(Z_4, \cdot)$. We find that addition is commutative in $(Z_4, +)$ and that multiplication is commutative in $(Z_4, \cdot)$. We have observed these properties previously.

Let us now examine a property which may be new to you. First, check that the following equations are true statements in

$(Z_4, +)$:

$$0 + 0 = 0 \tag{1}$$

$$2 + 2 = 0 \tag{2}$$

$$1 + 3 = 0 \tag{3}$$

$$3 + 1 = 0 \tag{4}$$

In each of the above mathematical statements we have a pair of numbers whose sum is 0 in $(Z_4, +)$. Or, we could say that we have found pairs of numbers in Z_4 whose sum is the identity element 0 in $(Z_4, +)$. The numbers in such a pair, whose sum in $(Z_4, +)$ is the identity element, are called inverses of each other under addition in Z_4. The numbers 1 and 3 are inverses of each other since $1 + 3 = 0$ in $(Z_4, +)$. We also say that 1 is the inverse of 3 and 3 is the inverse of 1. Note that (1) shows that 0 is its own inverse and (2) shows that 2 is its own inverse. Each of (3) and (4) show that 1 and 3 are inverses of each other.

We can search for inverses in any arithmetic which has an identity element. Thus in $(Z_4, \cdot)$ we will say that a pair of numbers are inverses if their product is the identity element 1. The inverse of 3 is easily obtained by examining the table for $(Z_4, \cdot)$. We simply go along the row headed by "3" (the last row in the table) until we find the identity element "1." Then the number heading the column which contains this "1" is the inverse of 3. We find a "3." Thus 3 is its own inverse. If we seek the inverse of 2, we go along the row headed by "2" until we find the identity element "1." But no "1" appears in this row. Thus 2 has no inverse under multiplication in Z_4.

Let us use the symbol "-3" to name the inverse of 3 under addition in Z_4. The symbol is read "the additive inverse of 3." Earlier we said that the inverse of 3 is 1 under addition in Z_4. Thus "-3" and "1" are different names for the same number in $(Z_4,+)$. Because of this, another way of writing "$3 + 1 = 0$" would be "$3 + (-3) = 0$" The following examples show some uses of this new symbol.

Example 1. In $(Z_4,+)$ we have $2 + (-2) = 0$. To justify this statement examine the definition of "-2."

Example 2. In $(Z_4,+)$ we have $-1 + (-2) = 1$. To justify this statement we first note that $-1 = 3$ and $-2 = 2$ in $(Z_4,+)$. Why? Convince yourself that "$3 + 2 = 1$" is a true statement in $(Z_4,+)$. If we replace "3" by "-1" and if we replace "2" by "-2," then we conclude that $-1 + (-2) = 1$ in $(Z_4,+)$.

Example 3. $-(1 + 2) = -1 + (-2)$ in $(Z_4,+)$. The symbol "$-(1 + 2)$" means the additive inverse of $1 + 2$ or, what is the same thing, the additive inverse of 3. We have then $-(1 + 2) = -3 = 1$. We saw in Example 2 that $-1 + (-2) = 1$. Since both $-(1 + 2)$ and $-1 + (-2)$ are equal to 1 we conclude that $-(1 + 2) = -1 + (-2)$.

Example 4. "$3 - 1$" and "$3 + (-1)$" name the same number in Z_4. We know that $3 - 1 = 2$. Also $3 + (-1) = 3 + 3 = 2$. Since both $3 - 1$ and $3 + (-1)$ are equal to 2, we have $3 - 1 = 3 + (-1)$.

Now that we have a special symbol to represent additive inverses in $(Z_4, +)$, let us select a symbol to represent inverses in $(Z_4, \cdot)$. Since we know that in $(Z_4, \cdot)$ $3 \cdot \frac{1}{3} = 1$, let us select the symbol " $\frac{1}{3}$ " to designate the inverse of 3 under multiplication in Z_4. We read this symbol as "1 over 3." Since 3 is its own inverse in $(Z_4, \cdot)$ it is clear that $\frac{1}{3} = 3$ in $(Z_4, \cdot)$. Similarly $\frac{1}{1} = 1$ in $(Z_4, \cdot)$. The following examples illustrate some uses of this new symbol.

Example 1. In $(Z_4, \cdot)$ we have $\frac{1}{3} \cdot 2 = 2$. Note that $\frac{1}{3}$, the the multiplicative inverse of 3 under multiplication in Z_4, is 3 in $(Z_4, \cdot)$. In short, $\frac{1}{3} = 3$ in $(Z_4, \cdot)$. Furthermore, $3 \cdot 2 = 2$ is a true statement in $(Z_4, \cdot)$. If we replace "3" in this statement with " $\frac{1}{3}$," then we conclude that $\frac{1}{3} \cdot 2 = 2$ in $(Z_4, \cdot)$.

Example 2. The solution set for $x \cdot \frac{1}{3} = 1$ is $\{3\}$ in $(Z_4, \cdot)$.

Example 3. The symbol " $\frac{1}{0}$ " does not name any number in Z_4. This is true because 0 does not have an inverse under multiplication in Z_4.

1.21 Exercises

Note: Unless otherwise stated, all of the exercises in this section should be considered using Z_4 arithmetic.

1. Using your addition table for $(Z_4, +)$ determine the additive inverse of:

(a) 2 (d) 3

(b) 1 (e) 4

(c) 0 (f) -2

2. Using your table for $(Z_5, \cdot)$ determine the multiplicative inverse of:

(a) 2 (d) 3

(b) 4 (e) 0

(c) 1 (f) $\frac{1}{2}$

3. Find the simplest names for the following:

(a) in $(Z_5, +)$ (b) in $(Z_5, \cdot)$

(1) -1 (1) $\frac{1}{2}$

(2) -4 (2) $\frac{1}{1}$

(3) 0 (3) $\frac{1}{4}$

(4) -2 (4) $\frac{1}{3}$

(5) -3 (5) $\frac{1}{0}$

4. Compute the following in $(Z_5, +)$:

(a) $3 + (-2)$ (d) $-3 + 3$

(b) $-4 + 1$ (e) $-2 + (-4)$

(c) $-1 + (-3)$ (f) $0 + 0$

5. Compute the following in $(Z_5, \cdot)$:

(a) $\frac{1}{2} \cdot 4$ (c) $\frac{1}{3} \cdot \frac{1}{2}$

(b) $\frac{1}{3} \cdot 3$ (d) $\frac{1}{2} \cdot \frac{1}{4}$

6. Solve the following open sentences:

   (a) in $(Z_5, +)$

   (1) $3 + (-3) = x$

   (2) $-4 + 1 = y$

   (3) $-2 + z = 4$

   (b) in $(Z_5, \cdot)$

   (1) $3 \cdot \frac{1}{3} = x$

   (2) $\frac{1}{2} \cdot y = 3$

   (3) $z \cdot \frac{1}{4} = \frac{1}{3}$

7. Note in $(Z_5, +)$ that the symbol $-(-2)$ means the additive inverse of the additive inverse of 2.

   (a) If we replace the name "-2" in the above symbol with the name "3," then what number in Z_5 do we have?

   (b) What numbers in Z_5 do the following represent?

   (1) $-(-4)$

   (2) $-(-1)$

   (3) $-(-3)$

   (4) $-(-0)$

   (c) What is the additive inverse of the additive inverse of 3?

   (d) Form a generalization from examining (a), (b) and (c) above.

8. (a) Explain why the symbol " $\frac{1}{2}$ " does not name a number in Z_4.

   (b) Does " $\frac{1}{0}$ " name a number in W?

   (c) Which whole numbers have inverses in $(W, +)$?

   (d) Which whole numbers have inverses in $(W, \cdot)$?

9. The additive inverse of $(3 + 1)$ in $(Z_5, +)$ can be named "$-(3 + 1)$" or "-4."

   (a) In $(Z_5, +)$ give the simplest name for each of the following:

   (1) $-(1 + 3)$

   (2) $-1 + (-3)$

   (3) $-(2 + 4)$

   (4) $-2 + (-4)$

   (5) $-(0 + 2)$

   (6) $-0 + (-2)$

(b) What is the additive inverse of the sum of 2 and 3 in $(Z_8,+)$?

(c) What is the sum of the additive inverse of 2 and the additive inverse of 3 in $(Z_8,+)$?

(d) Form a generalization from examining 9 (a), (b), and (c) above.

*10. After you have solved 9 (d) experiment with multiplicative inverses in $(Z_8,\cdot)$. Can you find evidence for a corresponding generalization about multiplicative inverses?

*11. Two examples of true statements about $(Z_8,+)$ are the following:

(i) If x̲ and y̲ are elements of Z_8, $x + y = y + x$ in $(Z_8,+)$.

(ii) If "$-x$" means the additive inverse of x̲ then $x + (-x) = 0$ in $(Z_8,+)$, where x̲ is any element in Z_8.

Explain why the following sentences are true or not true for every x̲ and y̲ in Z_8.

(a) In $(Z_8,+)$, the difference $x - y$ is the same as the sum of $x + (-y)$. (To subtract y̲ from x̲, we may add to x̲ the additive inverse of y̲. That is $x - y = x + (-y)$.)

(b) In Z_8 arithmetic $x - y = -(y - x)$. (The additive inverse of $y - x$ is $x - y$.)

(c) In $(Z_8,\cdot)$ $x \cdot \frac{1}{x} = 1$, for all x̲ in Z_8 except $x = 0$.

12. An important property of $(W,\cdot)$ is the following:

Let x and y be any elements in W. If $x \cdot y = 0$ then $x = 0$ or $y = 0$. Explain why there is, or is not, a corresponding property in the following:

(a) (Z[illegible],·) (b) (Z[illegible],·)

1.22 The Associative and Distributive Properties

The following is a famous grammatical puzzle. "Can you punctuate the string of words in the box so that a correct English sentence is formed?"

John where James had had had had had had had

It turns out we can solve the above puzzle by using the grammatical symbols,. " " Try it!

In mathematics we also make use of symbols which, like grammatical symbols, allow us to write expressions which are clear and correct. The most common "grammatical" symbols used in mathematics are parentheses. Consider the two expressions given below where addition is to be performed in $(Z_4,+)$.

$$(2 + 3) + 1 \qquad (1)$$

$$2 + (3 + 1) \qquad (2)$$

In (1) we see that "2 + 3" has been enclosed in parentheses. The parentheses are used to "signal" that we should consider "2 + 3" as naming a single number. Since we are to perform addition in $(Z_4,+)$ this number is 1. Thus, we have

$$(2 + 3) + 1 = 1 + 1 = 2 \text{ in } (Z_4,+).$$

In (2) the parentheses are used to signal that we should consider "3 + 1" as naming a single number. Thus, we have

$$2 + (3 + 1) = 2 + 0 = 2 \text{ in } (Z_4,+).$$

We note that the result of adding the numbers in (1) was the same as the result of adding the numbers in (2). A question that we might ask is the following: If $\underline{a}$, $\underline{b}$, and $\underline{c}$ are any triple of numbers in Z_4, will it always be true that

$$a + (b + c) = (a + b) + c \text{ in } (Z_4,+)?$$

If the answer to the question is "Yes," for all triples of numbers in Z_4, then we say that addition is associative in $(Z_4,+)$.

Next let us examine the same triple of numbers in Z_4 but ask if subtraction is associative in $(Z_4,+)$.

$$(2 - 3) - 1 \qquad (3)$$

$$2 - (3 - 1) \qquad (4)$$

We are asking if the result of computing (3) in $(Z_4,+)$ is the same as the result of computing (4) in $(Z_4,+)$. Because of the parentheses in (3) we first compute 2 - 3 in $(Z_4,+)$. Carrying out the subtractions in (3) we have

$$(2 - 3) - 1 = 3 - 1 = 2 \text{ in } (Z_4,+)$$

However from (4) we have

$$2 - (3 - 1) = 2 - 2 = 0$$

Thus $2 - (3 - 1) \neq (2 - 3) - 1$ in $(Z_4,+)$

and we say that subtraction is not associative in $(Z_4,+)$: not associative, because it failed for at least one triple of numbers.

Up to now when we sought out such properties as "commutativity" or "associativity" we confined ourselves to a single operation on a set of numbers. The next property that we shall investigate has a different role to play. It deals with two operations on a set of numbers. Let us consider the following two expressions where addition is to be performed in $(Z_4,+)$ and multiplication is to be performed in $(Z_4,\cdot)$. We shall indicate that we are working with one set and two operations by writing "$(Z_4,+,\cdot)$."

$2 \cdot (3 + 1)$ (5)

$(2 \cdot 3) + (2 \cdot 1)$ (6)

Again, the parentheses "signal" how our computations should proceed. In (5), since $3 + 1$ is considered as a single number, we compute as follows:

$$2 \cdot (3 + 1) = 2 \cdot 0 = 0 \text{ in } (Z_4, +, \cdot).$$

We compute (6) as follows:

$$(2 \cdot 3) + (2 \cdot 1) = 2 + 2 = 0 \text{ in } (Z_4, +, \cdot).$$

Since the computation in (5) and (6) both resulted in 0, we conclude that

$$2 \cdot (3 + 1) = (2 \cdot 3) + (2 \cdot 1).$$

If for every triple of numbers a, b, and c in Z_4 it is true that

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \text{ in } (Z_4, +, \cdot),$$

then we say that multiplication is distributive over addition in $(Z_4, +, \cdot)$.

Let us examine the same triple of numbers in Z_4 but ask instead, "Is addition distributive over multiplication in $(Z_4, +, \cdot)$?" Here we must compute the following in $(Z_4, +, \cdot)$:

$2 + (3 \cdot 1)$ (7)

$(2 + 3) \cdot (2 + 1)$ (8)

In (7) we have $2 + (3 \cdot 1) = 2 + 3 = 1$ in $(Z_4, +, \cdot)$.
In (8) we have $(2 + 3) \cdot (2 + 1) = 1 \cdot 3 = 3$ in $(Z_4, +, \cdot)$.
We conclude that $2 + (3 \cdot 1) \neq (2 + 3) \cdot (2 + 1)$ and that addition is not distributive over multiplication in $(Z_4, +, \cdot)$.

Compute the following in $(Z_4,+,\cdot)$:

$2 \cdot (3 - 1)$ (9)

$(2 \cdot 3) - (2 \cdot 1)$ (10)

From (9) we have $2 \cdot (3 - 1) = 2 \cdot 2 = 0$

From (10) we have $(2 \cdot 3) - (2 \cdot 1) = 2 - 2 = 0$

Note: Although $2 \cdot (3 - 1) = (2 \cdot 3) - (2 \cdot 1)$ in $(Z_4,+,\cdot)$ we cannot yet conclude that multiplication is distributive over subtraction in Z_4. Recall that the property must hold for all triples of numbers in Z_4. Experiment further with other triples of numbers and make a conjecture concerning the existence of a distributive property of multiplication over subtraction in Z_4.

1.23 Exercises

1. Compute the following in $(Z_5,+)$:
   - (a) $(2 + 4) + 3$
   - (b) $2 + (4 + 3)$
   - (c) $1 + (2 + 3)$
   - (d) $(1 + 2) + 3$
   - (e) $(3 + 0) + 4$
   - (f) $3 + (0 + 4)$
   - (g) $4 + (3 + 3)$
   - (h) $(4 + 3) + 3$
2. Compute the following in $(Z_5,\cdot)$:
   - (a) $(2 \cdot 4) \cdot 3$
   - (b) $2 \cdot (4 \cdot 3)$
   - (c) $1 \cdot (2 \cdot 3)$
   - (d) $(1 \cdot 2) \cdot 3$
   - (e) $(3 \cdot 0) \cdot 4$
   - (f) $3 \cdot (0 \cdot 4)$
   - (g) $4 \cdot (3 \cdot 3)$
   - (h) $(4 \cdot 3) \cdot 3$
3. Compute the following in $(Z_5,+,\cdot)$:
   - (a) $2 \cdot (4 + 3)$
   - (b) $(2 \cdot 4) + (2 \cdot 3)$
   - (e) $3 \cdot (0 + 4)$
   - (f) $(3 \cdot 0) + (3 \cdot 4)$

(c) $1 \cdot (2 + 3)$ (g) $4 \cdot (3 + 3)$

(d) $(1 \cdot 2) + (1 \cdot 3)$ (h) $(4 \cdot 3) + (4 \cdot 3)$

4. Compute the following in $(Z_6, +, \cdot)$:

(a) $-4 \cdot (3 + (-3))$ (c) $-2 \cdot (4 - (-3))$

(b) $(-4 \cdot 3) + (-4 \cdot -3)$ (d) $(-2 \cdot 4) - (-2 \cdot -3)$

5. (a) Is multiplication distributive over subtraction in Z_6?

(b) Is division distributive over subtraction in Z_6?

*6. The property $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ is more properly referred to as a "left hand" distributive property of multiplication over addition. Is there a corresponding "right hand" distributive property namely, $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$, in $(Z_6, +, \cdot)$?

*7. Assume that for all a, b, and c in Z_6 that $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ in $(Z_6, +, \cdot)$.

(a) Using known properties of $(Z_6, +)$ and $(Z_6, \cdot)$ can you prove that $a \cdot (b + c) = (c \cdot a) + (b \cdot a)$?

(b) Would $a \cdot (b + c) = (c \cdot a) + (b \cdot a)$ "hold" in $(Z_m, +, \cdot)$ where a, b, and c are any elements of Z_m and m is a whole number greater than 1?

1.24 Summary

1. In this chapter we studied a collection of finite sets called "clock numbers." We found that there were applications of these sets dealing with dials, rotations, codes, etc.

2. We defined addition, subtraction, multiplication, and division on these finite sets.

   (a) We found there were similarities between clock number arithmetic and whole number arithmetic: In (W,+) and in the clock arithmetics we studied such as $(Z_4,+)$ and $(Z_5,+)$, we saw that addition is an unrestricted operation. There are corresponding properties for both whole number arithmetic and the clock arithmetics dealing with an identity element for multiplication, an identity element for addition, commutative properties for addition and multiplication, associative properties for both addition and multiplication, a distributive property of multiplication over addition. Division by the additive identity is not defined in either whole number or clock arithmetic.

   (b) We found there were differences between clock number arithmetic and whole number arithmetic: The sets Z_4, Z_5, Z_{12}, etc. are finite, whereas the set W is infinite. Subtraction is a restricted operation on the set W but not on Z_4 or Z_7.

   Every element in Z_m has an additive inverse whereas only 0 in W had an additive inverse. The solution sets for corresponding open sentences in whole number arithmetic and clock arithmetic can differ greatly. For example: The open sentence $3 \cdot x = 3$ has the solution set $\{1\}$ in $(W,\cdot)$ whereas the corresponding open sentence in $(Z_6,\cdot)$ has the solution set $\{1,3,5\}$.

3. New terms were introduced and used. Among these were "statement," "variable," "open sentence," "is less than," "is greater than," "equation," "inequation," "solution set," "empty set," "additive inverse," "multiplicative inverse," "commutativity," "associativity," "distributivity." Check over the above terms to see if you understand what they mean. Where there is doubt recheck the meanings given in the text.

4. As you continue to study mathematics many of the ideas and terms found in this chapter will be given precise definitions and meanings. In particular, Chapter 2 will explore the idea of "unrestricted operation" by considering many new and interesting operations on sets.

1.25 Review Questions

Make up tables for $(Z_8,+)$ and $(Z_8,\cdot)$.

1. Compute the following in $(Z_8,+)$:

(a) $6 + 7$

(b) $5 + 3$

(c) $(7 + 7) + 6$

(d) $7 + (7 + 6)$

(e) $3 + (7 + -5)$

(f) $-3 + (-5 + (-3))$

2. Compute the following in $(Z_8,\cdot)$:

(a) $6 \cdot 7$

(b) $2 \cdot 4$

(c) $3 \cdot (4 \cdot 5)$

(d) $(3 \cdot 4) \cdot 5$

(e) $\frac{1}{3} \cdot (\frac{1}{5} \cdot \frac{1}{7})$

(f) $\frac{1}{5} \cdot (3 \cdot 7)$

3. Compute the following in $(Z_8,+,\cdot)$:

   (a) $3 \cdot (7 + 5)$ (c) $6 \cdot (7 - 5)$
   (b) $(3 \cdot 7) + (3 \cdot 5)$ (d) $6 \cdot (7 + (-5))$

4. Let a and b be any elements of Z_8. Explain why the following is true or false in $(Z_8,\cdot)$.

   If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

5. List all the elements of Z_8 and their corresponding inverses in $(Z_8,+)$.

6. List all the elements of Z_8 and their corresponding inverses in $(Z_8,\cdot)$.

7. Solve the following open sentences in $(Z_8,+,\cdot)$.

   (a) $3 + x = 5$ (f) $4 \cdot z = 0$
   (b) $y + 2 = 6$ (g) $3 \cdot x = 7$
   (c) $3 \cdot x = 5$ (h) $-5 \cdot x = 7$
   (d) $2 \cdot y = 0$ (i) $2 \cdot y = 3$
   (e) $3 - 7 = x$ (j) $4 \div (3 + 5) = x$

8. If today is Sunday, what day of the week is 1000 days from today? Explain your answer.

9. A circular bus route has 20 stops each 5 minutes apart. Which stop should the relief bus driver go to after the bus has been out seven and one quarter hours? Call the place where the route begins "stop 0," and call the first stop after this "stop 1," etc.

CHAPTER 2

SETS AND OPERATIONS

2.1 Ordered Pairs of Numbers and Assignments

Suppose you were given a pair of numbers, say 6 and 2, and were asked to assign a third number to this pair. Such an instruction might seem unclear, and indeed there are an endless number of answers that could be given. For example, one person might assign the number 8, since 6 + 2 = 8. We could show this assignment simply by writing

$$(6,2) \longrightarrow 8$$

to indicate that the pair of numbers (6,2) yields the number 8 if one is thinking of addition.

Another person, given the pair of numbers (6,2), might write

$$(6,2) \longrightarrow 3$$

and say that the pair (6,2) yields the number 3. We would probably guess that such a person is thinking of division, and we could write "$6 \div 2 = 3$."

If we were given the pair (2,6) and thought of division, we would write

$$(2,6) \longrightarrow \frac{1}{3}$$

since $2 \div 6 = \frac{1}{3}$. Thus, the pair (2,6) does not produce the same number as the pair (6,2). The order of the numbers in the pair is important. For this reason, we speak of an ordered pair of numbers. In the ordered pair (6,2), 6 is the first component of the pair and 2 the second.

Question: Are the ordered pairs (6,2) and (2,6) assigned the same number if one is thinking of addition?

Below are several ordered pairs of numbers. Each pair has been assigned a third number. In each case, tell how you think the third number was assigned.

$$(3,2) \longrightarrow 5$$
$$(3,2) \longrightarrow 1$$
$$(2\tfrac{1}{2},3\tfrac{1}{2}) \longrightarrow 6$$
$$(\tfrac{1}{4},\tfrac{1}{2}) \longrightarrow \tfrac{1}{8}$$
$$(.50,.25) \longrightarrow .125$$
$$(3,2) \longrightarrow 9$$
$$(2,3) \longrightarrow 8$$

The last two assignments in the above list result from raising a number to a power. Given the ordered pair (3,2), raising 3 to the power 2 means that we are to use 3 as a factor twice-- that is, 3 x 3, obtaining 9. This is often written as

$3^2 = 9$ (9 is a power of 3; specifically, 9 is the second power of 3.)

Similarly, given the ordered pair (2,3), we may think of 2 raised to the power 3. This means that we are to use 2 as a factor 3 times. Thus,

$2^3 = 2 \times 2 \times 2 = 8$ (8 is the third power of 2.)

This explains the assignments $(3,2) \longrightarrow 9$ and $(2,3) \longrightarrow 8$. Clearly, if one is thinking of raising a number to a power, the ordered pairs (3,2) and (2,3) are not assigned the same number.

Questions: What number is assigned to the ordered pair (2,2) by the process of raising to a power? To (3,3)? (Notice that the same number may be used for both the first and second component of an ordered pair.)

It is often convenient to use a table (as we did in Chapter 1) to show numbers assigned to pairs of numbers. For example, at the left below is a table showing some of the assignments made if one thinks of addition. At the right is a table showing some assignments if one is thinking of raising to a power.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

to the power	1	2	3	4	5
1	1	1	1	1	1
2	2	4	⑧	16	32
3	3	⑨	27	81	243
4	4	16	64	256	1024
5	5	25	125	625	3125

Do you see how the entries in the tables were obtained? Notice that in the second table the entries "9" and "8" have been circled, emphasizing that (3,2) and (2,3) yield different results.

Question: Suppose a is some whole number. What number is assigned to the ordered pair (a,1) by the process of raising to a power? How is this shown in the table of powers above? What number is assigned to the ordered pair (1,a) by raising to a power? How is this shown in the table of powers?

Below is still another table showing assignments of numbers to pairs of numbers. These assignments should be familiar from your work in Chapter 1. Do you see how they were obtained?

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2.2 Exercises

1. Tell what number is assigned to the following ordered pairs by usual addition.

   (a) (5,0) (b) (0,5) (c) (6,6) (d) (218,365)
   (e) (365,218) (f) (750,250) (g) $(\frac{2}{3},\frac{1}{5})$
   (h) $(4\frac{1}{2},2\frac{3}{4})$ (i) (.83,.27)
   (j) (2000000,8000000)

2. (a) Working with whole numbers only, list all ordered pairs of whole numbers to which the number 5 is assigned by addition. (Remember that (a,b) and (b,a) are different ordered pairs.)

   (b) Again using whole numbers only, list all ordered pairs to which the number 1 is assigned by addition.

   (c) List all ordered pairs of whole numbers to which the number 0 is assigned by addition.

3. (a) List all ordered pairs of whole numbers to which 24 is assigned by multiplication.

(b) List all ordered pairs of whole numbers to which the number 13 is assigned by multiplication.

(c) List all ordered pairs of whole numbers to which 0 is assigned by multiplication.

4. Tell what number is assigned to the following ordered pairs by multiplication.

(a) (5,0) (b) (0,5) (c) (8,6) (d) (51,106)

(e) (106,51) (f) $(4\frac{1}{2}, 2\frac{3}{4})$ (g) $(\frac{2}{3}, \frac{3}{4})$ (h) $(1\frac{1}{4}, \frac{4}{5})$

(i) (.6,.5) (j) (.83,.27)

5. In the text, we explained raising a whole number to a power. 4^2 means 4 x 4 or 4 used as a factor 2 times. Often in mathematics, we use a raised dot "·" instead of an "x" to indicate multiplication. Thus, we may write $4^2 = 4 \cdot 4 = 16$. An expression such as "a^b" is read "a to the b power," and the number b is called an exponent. We are assuming that both a and b are whole numbers. With this in mind, tell what numbers the following name.

(a) 2^3 (b) 2^2 (c) 2^1 (d) 10^1 (e) 10^2

(f) 10^3 (g) 10^4 (h) 10^6 (i) 5^2 (j) 2^5

(k) 4^3 (l) 3^4 (m) 3^3 (n) 1^{546}

6. If we think of "raising to a power" as assigning numbers to ordered pairs of numbers, what number is assigned to the following pairs by "raising to a power?" Remember that we take the second number as the exponent.

(a) (3,4) (b) (4,3) (c) (4,2) (d) (2,4)

(e) (3,5) (f) (5,3)

7. (a) List all ordered pairs of whole numbers which are assigned the number 16 under "raising to a power."

(b) List all ordered pairs of whole numbers which are assigned the number 10 by "raising to a power."

8. We know that assignments of numbers to pairs can be shown by a table. Fill in all the cells in the following table for addition. (Notice in this case that the numbers are not listed in any particular order.)

+	5	682	17	8	0	1	1720
5							
682							
17							
8							
0							
1							
1720							

9. Copy the table in Exercise 8. Then fill in the cells by using multiplication instead of addition.

10. (a) In what cases do the ordered pairs of whole numbers (a,b) and (b,a) produce the same number under addition?

(b) In what cases do the ordered pairs of whole numbers (a,b) and (b,a) produce the same number under "raising to a power?"

11. In the list of expressions below, n represents a number.

$$3 \cdot (n^2)$$

$$[3 \cdot (n^2)] + 2$$

$$[3 \cdot (n^2)] - 2$$

$$\frac{1}{2 \cdot (n^2)}$$

$$\left[\frac{1}{2 \cdot (n^2)}\right] + 8$$

$$[2 \cdot (n^3)] + 5$$

(a) Find what number each expression represents if $n = 0$.

(b) Find what number each expression represents if $n = 2$.

(c) Find what number each expression represents if $n = 5$.

(d) Find what number each expression represents if $n = 10$.

(e) Find what number each expression represents if $n = 100$.

(f) Find what number each expression represents if $n = \frac{1}{2}$.

*12. In this problem, we look at another way of assigning a number to an ordered pair of numbers, specifically an ordered pair of natural numbers (the whole numbers except zero). Consider the ordered pair (24,16) of whole numbers. The set of whole numbers which divide 24 is

$$\{1, 2, 3, 4, 6, 8, 12, 24\}$$

The set of whole numbers which divide 16 is

$$\{1, 2, 4, 8, 16\}$$

Notice there are some numbers (1, 2, 4, and 8) which divide both 24 and 16. Of these, 8 is the greatest. Therefore, we call 8 the greatest common divisor of 24 and 16. If we agree to assign the greatest common divisor to the ordered pair (24,16), we will write

$$(24,16) \longrightarrow 8.$$

Under this same scheme, we would make the assignment

$$(12,18) \longrightarrow 6.$$ (Do you see why?)

Use the "greatest common divisor" idea to make assignments to the following ordered pairs:

(a) (6,8) (b) (6,12) (c) (10,15) (d) (100,200)
(e) (21,45) (f) (7,9) (g) (1,10) (h) (4,4)
(i) (21,42) (j) (42,21)

2.3 What Is An Operation?

You know from arithmetic that given an ordered pair of whole numbers, we can assign a number called their sum. For example,

$$(3,5) \longrightarrow 8.$$

Addition assigns to every ordered pair in W one and only one whole number which is their sum. We call addition an operation on W and refer to the ordered pair (W,+) as an operational system.

There are many interesting operations on the set of whole numbers. As an example, consider the "maximizing" operation. To illustrate the way the maximizing operation assigns whole numbers to ordered pairs of whole numbers, consider the ordered pair (6,2). Of the two numbers making up the pair, 6 is the larger. Therefore, we assign 6 to the pair.

$$(6,2) \longrightarrow 6.$$

As another illustration, under this operation we assign 10 to the ordered pair (3,10). To every ordered pair (a,b), we assign the larger of the two numbers, a and b. It is possible that a and b may be the same number, as in the pair (3,3). In such a case, we shall simply assign the number itself to the pair.

$$(3,3) \longrightarrow 3.$$

Do you see that here again there is no doubt about the number to be assigned? Every ordered pair of whole numbers is assigned one and only one whole number. Therefore, like addition, maximizing is an operation on the set of whole numbers, and (W,max) is an operational system.

Now suppose we consider "taking the average" of two whole numbers. (The average we are speaking of here is more properly called the arithmetic mean.) The average of 5 and 13 is 9, the average of 6 and 10 is 8. If we write these as assignments, we have the following:

$$(6,10) \longrightarrow 8;$$

$$(5,13) \longrightarrow 9.$$

Now take a pair such as (5,8). There is no whole number which is the average of 5 and 8. You may know that $6\frac{1}{2}$ is the average here, but $6\frac{1}{2}$ is not a whole number. If we are working only with whole numbers, there is no number to be assigned to the pair (5,8). Therefore averaging is not an operation on the whole numbers, because we have a pair of whole numbers to which no assignment can be made. However, averaging is an operation on the set of numbers of arithmetic, since the average of any two such numbers can be computed. (See Exercise 17.)

Let us look at each of these examples again:

Addition: Here we have $(3,5) \longrightarrow 8$. Since we have the well known symbol "+" for addition, we could just as well write

$$(3,5) \longrightarrow 3 + 5.$$

In this case it is easier to write "8," but suppose we want to talk about any pair of whole numbers. We might designate this pair as "(a,b)" and then write

$(a,b) \longrightarrow a + b$, for every whole number *a* and every whole number *b*

under the operation of addition.

Maximizing: In working with this operation on the whole numbers, we can write $(8,3) \longrightarrow 8$. If we use "max(a,b)" to mean the greater of the two numbers *a* and *b*, we can write

$(a,b) \longrightarrow \max(a,b)$, for every whole number *a* and every whole number *b*

under the operation of maximizing. In the case of addition, the symbol "+" is written between the symbols "a" and "b." We can also do this in the case of "max" and write

$(a,b) \longrightarrow a \max b$, for every whole number *a* and every whole number *b*.

Averaging: We do not have a symbol for the average of two numbers. But again we can invent one. Let us agree that "aVb" shall mean, "the average of the whole numbers *a* and *b*." Thus "6V8 = 7" is just another way of indicating the assignment

$$(6,8) \longrightarrow 7$$

if one is thinking of averaging. As we saw earlier, however, 5V8 is not a whole number, and V is not an operation on the whole numbers.

Question: Name five other ordered pairs (a,b) of whole numbers for which aVb is not a whole number.

Below are three tables showing the assignments for certain pairs of whole numbers under the "+" and "max" operations and for averaging. An important point to notice is that there are open cells in the table for V (averaging). The fact that these cells are open emphasizes once again why averaging is not an operation on the whole numbers; there just are not any whole numbers which properly go in these cells.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

max	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
2	2	2	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

V	0	1	2	3	4
0	0		1		2
1		1		2	
2	1		2		3
3		2		3	
4	2.		3		4

We have seen three symbols "+," "max," and "V" used to denote schemes for assigning whole numbers to ordered pairs of whole numbers. V is not an operation since aVb is not a whole number for every ordered pair (a,b) of whole numbers. On the other hand, + and max are operations since a + b and a max b are, for every pair (a,b), unique whole numbers. These examples lead us to a general definition of an operation on the set of whole numbers.

Definition: Let * be a scheme for assigning numbers to ordered pairs of whole numbers. If * assigns to each ordered pair (a,b) of whole numbers one and only one whole number then * is a binary operation on the set of whole numbers.

The word "binary" in this definition is worth some attention. The prefix "bi-" is associated with the idea of a pair, or two things (think of "bicycle" and "biped," for instance). Thus, a binary operation is one which assigns a number to a pair of numbers. Suppose c is the whole number assigned to the ordered pair (a,b) by operation *. Then we write

$$a * b = c.$$

(If you think again about addition of whole numbers, you will see that this is what we have always done there. For instance, + assigns the number 10 to the ordered pair (6,4), and we write "6 + 4 = 10.")

The notion of operation is a very general one and may be applied to any set, not just the set W of whole numbers. As one example, consider again the operation of maximizing. This time we shall work with the set S = {1,2,3,4,5}, which is a finite subset of the whole numbers. The table below shows a max b for the ordered pairs (a,b) of numbers in S.

max	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	4	5
3	3	3	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

Notice from this table that for every ordered pair (a,b) of numbers in S, a max b is a number in S. Therefore, "max" is an operation on the set S as well as on the set W of whole numbers.

The last example suggests a more general definition of operation.

Definition: A binary operation * on a set S is an assignment which assigns to each ordered pair (a,b) of elements in S, one and only one element c in S.

The definition says essentially the same thing as the earlier one, except that this time we did not restrict ourselves to the set W of whole numbers. In fact, the elements of S need not be "numbers" at all! (See Exercise 16.) We denote the operational system consisting of the set S and the operation * by "(S,*)."

2.4 Exercises

1. What number does each of the following ordered pairs of whole numbers produce under the operation of maximizing discussed in the text? (When "a" is used, it is meant to be a whole number.)

   (a) (0,0) (b) (0,1) (c) (1,0) (d) (5,15) (e) (15,5)

   (f) (30,100) (g) (2010,2008) (h) (999,1000)

   (i) (a,a + 1) (j) (a,1 · a) (k) (a,0).

2. Evaluate each of the following in W:

   (a) 6 max 2 (b) 6 + 2 (c) 6 · 2 (d) 6 - 2 (e) 6 ÷ 2

   (f) 588 + 92 (g) 1001 - 865 (h) 88 x 97 (i) 483 ÷ 3

   (j) 82 x 10,000

3. Is subtraction an operation on the set of whole numbers? (Hint: Does subtraction assign a whole number to the ordered pair (2,5)?)

4. Is division an operation on the set of whole numbers?

5. Suppose we decide to assign to every ordered pair of whole numbers (a,b) a whole number which divides both a and b. Explain why such a procedure does not define an operation. (Hint: Consider the pair (8,12). Is there more than one possible assignment?)

6. In this problem, let "a*b" mean "the greatest common divisor of a and b." (See Exercise 12 of Section 2.2.)

   (a) Is * an operation on the set

   $$W = \{0, 1, 2, 3, 4, \ldots\} \ ?$$

   (b) Is * an operation on the set

   $$N = \{1, 2, 3, 4, 5, \ldots\} \ ?$$

7. In this problem, we shall consider a new way of assigning a number to an ordered pair of natural numbers (the whole numbers except zero). To explain it, we shall use the ordered pair (6,8). Now 1 x 6 = 6, 2 x 6 = 12, 3 x 6 = 18, etc. Therefore 6, 12, 18, etc., are called multiples of 6. The list of multiples of 6 may be indicated as follows:

   6, 12, 18, 24, 30, 36, 42, 48, 54, ...

   In the same way, the multiples of 8 may be shown in the following way:

   8, 16, 24, 32, 40, 48, 56, 64, ...

   Of course, 6 and 8 have some multiples in common, such as 24, 48, 96, etc. Of these, 24 is the smallest, and we shall call it the least common multiple of 6 and 8. In this problem, let us use

   lcm(a,b)

to mean "the least common multiple of a and b." For example lcm(8,6) = 24, lcm(10,15) = 30. Do you see why? Evaluate the following:

(a) lcm(2,3) (f) lcm(1,5)
(b) lcm(5,10) (g) lcm(5,1)
(c) lcm(10,5) (h) lcm(100,1000)
(d) lcm(7,11) (i) lcm(90,70)
(e) lcm(11,7) (j) lcm(14,42)

8. Suppose we continue with the notion of least common multiple used in Problem 7. But this time let us work with the set W of whole numbers, which means that 0 is now included in our set. Thus, the set of multiples of 6 is

{0, 6, 12, 18, 24, 30, 36, ...}

Zero is included since 0 x 6 = 0. Similarly, the set of multiples of 8 is

{0, 8, 16, 24, 32, 40, 48, ...}

Now, with the understanding that "lcm(a,b)," means "least common multiple of a and b," where a and b are whole numbers, evaluate the following:

(a) lcm(2,3) (c) lcm(1,5)
(b) lcm(5,10) (d) lcm(5,1)

Do you see why lcm should be used for natural numbers and not for all whole numbers?

9. Answer the following questions on the basis of your work in Problems 7 and 8:

(a) Is lcm an operation on the set N of natural numbers?
(b) Is lcm an operation on the set W of whole numbers?

Be prepared to defend your answers.

10. Consider the set S = {0, 1} which is a finite set containing exactly two numbers.

    (a) Is ordinary addition an operation on set S? Construct a table showing all possible sums.

    (b) Is ordinary multiplication an operation on set S? Construct a table showing all possible products.

11. The set of even whole numbers is indicated below:

    {0, 2, 4, 6, 8, 10, ...}

    (a) Is addition an operation on the set of even whole numbers?

    (b) Is multiplication an operation on the set of even whole numbers?

    (c) Is raising to a power an operation on the set of even whole numbers?

12. The set of odd whole numbers is indicated below:

    {1, 3, 5, 7, 9, 11, ...}

    (a) Is addition an operation on the set of odd whole numbers?

    (b) Is multiplication an operation on the set of odd whole numbers?

    (c) Is raising to a power an operation on the set of odd whole numbers?

13. In Chapter 1 we worked with some finite systems. In this problem, we shall use the system $(Z_2, +)$. (A physical model for this system is furnished by a clock face with numerals "0" and "1.")

(a) Construct a table for $(Z_2,+)$.

(b) According to the definition of operation in Section 2.3, is + an operation on the set Z_2? Why or why not?

14. Let S be a set that has two elements, a and b. That is, S = {a, b}. We don't know what "things" a and b are, but suppose we are told that a is assigned to the ordered pair (a,a) and to the ordered pair (b,b) and b is assigned to the ordered pair (b,a) and to the ordered pair (a,b). These assignments are displayed in the table below:

	a	b
a	a	b
b	b	a

Does this table define an operation on the set (a,b)? Compare the table to that in part (a) of Problem 13. Do you see any similarities?

15. In the following "assignment" table, "a," "b," and "c" denote three different objects.

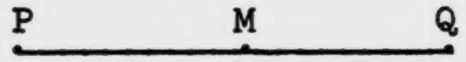
	a	b
a	a	b
b	b	c

(a) Does this table define an operation on the set {a,b}?

(b) Does the table define an operation on the set {a,b,c}?

16. Although we have not yet talked about geometry in this course, you probably have some idea of what a point is.

Given two points, you can find the point midway between them. This point is called the midpoint of the two given points. For example, P and Q are two points in the drawing below, and M is their midpoint.



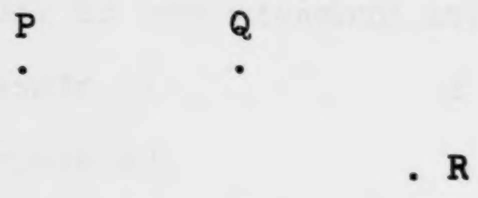
(a) Given an ordered pair of points (P,Q), where P and Q are different points, do P and Q have one and only one midpoint?

(b) If P and Q are different points, is the midpoint assigned to the ordered pair (P,Q) the same point as is assigned to the ordered pair (Q,P)?

(c) What midpoint would you assign to the ordered pair (Q,Q)?

(d) Consider the set of all ordered pairs of points. If mid(P,Q) means "the midpoint of P and Q," is mid an operation on the set of all points?

(e) If P, Q and R are three points as below, locate mid(mid(P,Q),R).



Is mid(mid(P,Q),R) the same point as mid(P,mid(Q,R))?

17. While averaging is not an operation on the set W of whole numbers, it is an operation on the set of numbers of arithmetic. Compute the following, where "V" means the assignment is to be the average of the two numbers making up

the pair.

(a) $\frac{1}{2}$ V $\frac{1}{4}$ (b) $2\frac{3}{4}$ V $5\frac{1}{8}$ (c) 2.8 V 6.4

(d) $10\frac{1}{2}$ V $10\frac{1}{2}$ (e) .624 V .875 (f) $9\frac{1}{2}$ V $10\frac{1}{4}$

2.5 Computations with Operations

You know by now what is meant by a binary operation on a set. You have seen that the symbol "*" is often used for an operation. (We have used special symbols such as "+" and "max" also.) In fact, any symbol at all may be used for a particular operation, as long as it is clear to what operation the symbol refers. In this and following sections, we are going to work with several different operations, and it would be troublesome to invent a new symbol for each one of them. On the other hand, we cannot use "*" for all of them. Therefore, we shall make use of subscripts, and denote the operations by symbols such as

$$*_1, *_2, *_3, \text{etc.}$$

Now let us define six different operations, some of them familiar and others probably new to you.

$*_1$: $a *_1 b = a \cdot b$	In other words, the $*_1$ operation is ordinary multiplication of whole numbers. For example, $5 *_1 3 = 15$.
$*_2$: $a *_2 b = a + b$	For example, $5 *_2 3 = 8$.
$*_3$: $a *_3 b = a \max b$	For example, $5 *_3 3 = 5$.
$*_4$: $a *_4 b = a$	In other words, this operation

assigns to every ordered pair the first number of the pair. For example $5 *_4 3 = 5$, but $3 *_4 5 = 3$.

$*_5 :\ a *_5 b = 17$ — Notice that this operation assigns the same number to every pair.

$*_6 :\ a *_6 b = a^2 + b^2$ — For example $5 *_6 3 = 5^2 + 3^2 = 25 + 9 = 34$.

In order to see how to compute with these six operations, we look at some examples.

Example 1: Find $3 *_6 2$.

The $*_6$ operation assigns to every ordered pair (a,b) the number $a^2 + b^2$. In our example, a is 3 and b is 2. Therefore $3 *_6 2 = 3^2 + 2^2 = 9 + 4 = 13$.

Example 2: Find $(3 *_6 2) *_2 4$.

The fact that "$3 *_6 2$" has been enclosed in parentheses means that we are to consider this as a single number. And, from Example 1, we know that this number is 13. Hence we may write $(3 *_6 2) *_2 4 = 13 *_2 4$.

But the $*_2$ operation is ordinary addition of whole numbers; so $13 *_2 4 = 17$. Therefore, we have $(3 *_6 2) *_2 4 = 13 *_2 4 = 17$.

Example 3: Find $3*_6(2*_2 4)$.

Compare this with Example 2. Although the same numbers and the same operations are involved, the parentheses have been differently placed. In this example, we are to consider "$2*_2 4$" as a single number. Since $*_2$ is ordinary addition, this number is 6. Thus we have

$$\begin{aligned} 3*_6(2*_2 4) &= 3*_6 6 \\ &= 3^2 + 6^2 \qquad \text{(Rember how the operation is defined.)} \\ &= 9 + 36 \\ &= 45 \end{aligned}$$

We see that the results in Examples 2 and 3 are not the same. This points up the importance of parentheses in mathematical expressions.

Example 4: Find $((4*_4 7)*_6 2)*_1 10$.

This expression contains two different "signals" in the form of parentheses. When we have parentheses within parentheses, it is always understood that the innermost pair is to be dealt with first.

We begin with $(4*_4 7)$ which is to be taken as a single number. From the way in which the $*_4$ operation is defined, we know that $4*_4 7$ is 4, since 4 is the first number of the pair (4,7). So, we have for a first step:

$$\begin{aligned} ((4*_4 7)*_6 2)*_1 10 &= (4*_6 2)*_1 10 \\ &= 20*_1 10 \end{aligned}$$

"$4 *_6 2$" has been replaced by "20." Finally, we know that $20 *_1 10$ is 200. If all the steps are written together, we have the following:

$$\begin{aligned}((4 *_4 7) *_6 2) *_1 10 &= (4 *_6 2) *_1 10 \\ &= 20 *_1 10 \\ &= 200.\end{aligned}$$

Sometimes, but not always, when an expression involves more than one pair of parentheses, a pair of brackets may replace a pair of parentheses. For instance, the expression of Example 4 might be written

$$[(4 *_4 7) *_6 2] *_1 10.$$

In the following example, the steps have been listed without any additional explanation. Be sure that you can explain each step.

Example 5: Find $(4 *_2 7) *_4 ((3 *_1 2) *_3 5)$.

$$\begin{aligned}(4 *_2 7) *_4 ((3 *_1 2) *_3 5) &= 11 *_4 (6 *_3 5) \\ &= 11 *_4 6 \\ &= 11.\end{aligned}$$

2.6 Exercises

In problems 1 through 20, the operations are those defined in Section 2.5 of the text.

1. (a) $5 *_1 2 =$ (e) $5 *_5 2 =$

(b) $5 *_2 2 =$ (f) $5 *_6 2 =$

(c) $5 *_3 2 =$

(d) $5 *_4 2 =$

2. (a) $(7*_3 3)*_3 8 =$

(b) $7*_3 (3*_3 8) =$

(c) $2*_6 (3*_6 5) =$

(d) $(2*_6 3)*_6 5 =$

3. (a) $109*_3 111 =$

(b) $111*_3 109 =$

(c) $109*_4 111 =$

(d) $111*_4 109 =$

4. (a) $58*_4 32 =$

(b) $32*_4 58 =$

(c) $58*_5 32 =$

(d) $32*_5 58 =$

5. (a) $42*_1 1 =$

(b) $42*_2 0 =$

(c) $42*_3 0 =$

(d) $615*_1 1 =$

(e) $615*_2 0 =$

(f) $615*_3 0 =$

6. (a) $(3*_2 5)*_2 4 =$

(b) $3*_2 (5*_2 4) =$

(c) $3*_6 (1*_6 4) =$

(d) $(3*_6 1)*_6 4 =$

7. (a) $(7*_3 5)*_2 8 =$

(b) $7*_3 (5*_2 8) =$

8. (a) $5*_6 (2*_1 3) =$

(b) $(5*_6 2)*_1 3 =$

9. (a) $(420*_5 3)*_1 85 =$

(b) $420*_5 (3*_1 85) =$

10. $((14*_5 3)*_4 2)*_1 10 =$

11. $15*_3 ((3*_2 5)*_2 889) =$

12. $[(8*_3 10)*_5 15]*_1 87 =$

13. $((2*_6 3)*_6 4)*_6 5 =$

15. $3*_1 (5*_2 6) =$

16. $3*_2 (5*_1 6) =$

17. $[5*_1 (2*_2 3)]*_2 [5*_2 (2*_1 3] =$

18. $(8*_4 12)*_3 (8*_5 12) =$

19. $((((2*_1 2)*_2 2)*_3 2)*_4 2)*_5 2 =$

20. $(((((2*_1 2)*_2 2)*_3 2)*_4 2)*_5 2)*_6 2 =$

21. Parentheses are also important in expressions involving the ordinary computations in arithmetic, as the following problems illustrate.

    (a) $\frac{1}{2} \cdot (\frac{2}{3} + \frac{3}{4}) =$

    (b) $(\frac{1}{2} \cdot \frac{2}{3}) + (\frac{1}{2} \cdot \frac{3}{4}) =$

    (c) $[(6\frac{1}{2} \cdot 3) + 3] \cdot 10 =$

    (d) $(4\frac{1}{2} + 2\frac{2}{3}) + \frac{1}{2} =$

    (e) $4\frac{1}{2} \cdot (2\frac{2}{3} + \frac{1}{2}) =$

    (f) $\frac{1}{2} + (2 + \frac{1}{3}) =$

    (g) $(\frac{1}{2} + 2) + \frac{1}{3} =$

    (h) $(((2\frac{1}{3} \cdot 1\frac{1}{2}) + \frac{1}{2}) + \frac{3}{4}) - \frac{1}{2} =$

22. Make up an operation over the whole numbers, and call it $*_7$. (Caution: Be sure that it is an operation!) Then compute the following:

    (a) $8 *_7 15 =$

    (b) $15 *_7 8 =$

    (c) $5 *_7 (2 *_7 3) =$

    (d) $(5 *_7 2) *_7 3 =$

23. Consider the following expressions:

    $$2(n^3);\ (2n)^3$$

    They are not the same. The first one is often written as "$2n^3$," without parentheses.

    (a) If these expressions are in $(W, \cdot)$ are there any values of n for which

    $$2n^3 = (2n)^3?$$

    (b) What is your answer if the expressions are in $(Z_8, \cdot)$?

2.7 Open Sentences

Consider an open sentence in W such as

$$5 *_2 x = 8$$

The operation $*_2$, according to our definition, is ordinary addition of whole numbers. Therefore, the question posed (and it is an easy one) is this:

> Is there an ordered pair (5,x) to which 8 is assigned by the operation of addition?

The answer is obvious; x must be 3 in order for this assignment to be made. Therefore we say that 3 is a solution of the open sentence "$5 *_2 x = 8$." In this case, it is easy to see that 3 is the only solution. But some open sentences have more than one solution; so you must be careful when "solving" an open sentence that you indicate all the solutions, not just some of them.

Example 1: Solve $3 *_3 n = 3$ in W.

The $*_3$ operation assigns max(a,b) to every ordered pair (a,b). Therefore, the open sentence will be true if and only if max(3,n) = 3. But this in turn will be true if n is 0, 1, or 2. It will also be true if n is 3, since max(3,3) = 3. Do you see, however, that it will no longer be true if n is a whole number greater than 3? Therefore the solution set of the open sentence is

$$\{0, 1, 2, 3\}.$$

In this case, we have exactly four solutions.

Example 2: Solve $2 *_3 a = a$ in W

Under the $*_3$ operation, the assignment $(2,a) \rightarrow a$ means that a is the greater of the two numbers, 2 and a. Therefore, in order to make the statement true, a must be 2 or any number greater than 2. There are infinitely many solutions! The solution set is

$$\{2, 3, 4, 5, \ldots\}.$$

Example 3: Solve $a *_6 2 = 29$ in W.

From the definition of the operation $*_6$ we know that if this sentence is to be true, then $a^2 + 2^2$ must be 29. But $2^2 = 4$; so $a^2 + 4$ must be 29. Now, if $a^2 + 4$ is 29, do you see that a^2 must be 25?

However, 25 is not a solution; we are looking for a, not a^2. But of course if a^2 is 25, then a is 5.

The complete list of steps might be written as follows:

$$a^2 + 4 = 29$$

$$a^2 = 25$$

$$a = 5$$

Is 5 the only whole number solution?

2.8 Exercises

In 1-26 find the solutions of the open sentences using the indicated operations as defined in Section 2.5. If there is more than one solution, be sure to find all of them. Use only whole numbers.

1. $8*_2 a = 11$
2. $11*_2 a = 8$
3. $5*_1 a = 10$
4. $10*_1 a = 5$
5. $n*_2 81 = 103$
6. $n*_2 103 = 81$
7. $n*_1 17 = 187$
8. $n*_1 187 = 17$
9. $5*_3 a = 5$
10. $a*_3 6 = 6$
11. $42*_4 a = 21$
12. $a*_4 42 = 42$
13. $42*_4 a = 42$
14. $85*_5 a = 17$
15. $85*_5 a = 18$
16. $2*_6 a = 13$
17. $a*_6 2 = 13$
18. $3*_6 a = 25$
19. $3*_6 a = 30$
20. $5*_4 a = 10$
21. $n*_4 15 = 60$
22. $3*_5 n = 3$
23. $52*_3 n = 1$
24. $32*_1 n = 321$
25. $n*_2 32 = 321$
26. (a) $832*_1 a = 832$
    (b) $832*_2 a = 832$
    (c) $832*_3 a = 832$
    (d) $832^{*}{}_4 a = 832$
    (e) $832*_5 a = 832$
    (f) $832*_6 a = 832$
27. Before solving the following open sentences in W, it is important to understand the following: Suppose you are asked to solve the open sentence "a + a = 6," where "+" is ordinary addition. Since 3 + 3 = 6, 3 is a solution. Notice that "a" is used more than once in the sentence, and the same number must be used for each

"a" in the sentence. Thus, although $4 + 2 = 6$, this does not give us a solution to the sentence.

(a) $3*_1a = a*_13$ (d) $3*_4n = n*_43$

(b) $3*_2a = a*_23$ (e) $3*_5x = x*_53$

(c) $3*_3n = n*_33$ (f) $3*_6x = x*_63$

28. (a) $a*_6a = 72$ (f) $a*_1a = 2$

(b) $a*_1a = 25$ (g) $x*_6x = 17$

(c) $n*_1n = 24$ (h) $x*_3x = 5$

(d) $n*_2n = 242$ (i) $a*_4a = 5$

(e) $a*_2a = 243$ (j) $a*_5a = 5$

29. (a) $n*_1(n*_1n) = 8$ (d) $a*_6(a*_6a) = 68$

(b) $a*_5(a*_5a) = 17$ (e) $n*_4(n*_4n) = 108$

(c) $n*_3(n*_3n) = 23$ (f) $a*_2(a*_2a) = 9$

2.9 Properties of Operations

Referring to the operations defined in Section 2.5, tell what number each of the following expression names:

$5*_32$ $5*_42$

$2*_35$ $2_{34}5$

$8*_37$ $8*_47$

$7*_38$ $7*_48$

$15*_3100$ $15*_4100$

$100*_315$ $100*_415$

In the $*_3$ operation, does the order of the numbers affect the number produced? It is easy to see from the way $*_3$ was defined, that the ordered pair (a,b) will always produce the same number as the ordered pair (b,a). We may state this formally as follows:

For every whole number a and every whole number b

$$a *_3 b = b *_3 a.$$

This is a statement of the commutative property of $*_3$, and we say that $*_3$ is a commutative operation on W. (You will recall the use of the word "commutative" from Chapter 1.)

From the list above, we see at once that the $*_4$ operation is not commutative. This conclusion follows from the fact that $5 *_4 2 \neq 2 *_4 5$, even without looking at the rest of the examples. We say that "$5 *_4 2 \neq 2 *_4 5$" is a counterexample; that is, it is an example counter to (or against) the commutativity of $*_4$. It is often easy to show that some general statement is false simply by finding one counterexample.

Again referring to the operations of Section 2.5, tell what number each of the following expression names:

$(2 *_6 3) *_6 5$	$(2 *_4 3) *_4 5$
$2 *_6 (3 *_6 5)$	$2 *_4 (3 *_4 5)$
$4 *_6 (1 *_6 3)$	$4 *_4 (1 *_4 3)$
$(4 *_6 1) *_6 6$	$(4 *_4 1) *_4 3$
$(2 *_6 2) *_6 6$	$(2 *_4 2) *_4 6$
$2 *_6 (2 *_6 6)$	$2 *_4 (2 *_4 6)$

From these examples, we see, for instance, that $(2 *_4 3) *_4 5 = 2 *_4 (3 *_4 5)$; that is, the result is the same whether the last two numbers or the first two numbers are associated by parentheses. The same is true for the other examples using the $*_4$ operation. It is in fact true no matter what three numbers are selected. We may state this as follows:

For every whole number a, every whole number b, and every whole number c,

$$(a *_4 b) *_4 c = a *_4 (b *_4 c).$$

This is a statement of the associative property of $*_4$, and we say that $*_4$ is an associative operation on W.

Question: From the list above, can you find a counterexample showing that $*_8$ is not an associative operation? Next, tell what number each of the following expressions names:

$5 *_3 0$	$5 *_8 0$
$0 *_3 5$	$0 *_8 5$
$142 *_3 0$	$142 *_8 0$
$0 *_3 142$	$0 *_8 142$
$55 *_3 0$	$55 *_8 0$
$0 *_3 55$	$0 *_8 55$

How may the "behavior" of the number 0 under the $*_3$ operation be described? Do you see (from the way the $*_3$ operation was defined, not just from the illustrations above) that for any whole number a, $*_3$ assigns to the pair (a,0) the number a itself? It also assigns a to the pair (0,a). In other words, for any whole number a, $a *_3 0 = a$, and $0 *_3 a = a$. We often put these statements together in the following way:

For every whole number a, $a *_3 0 = 0 *_3 a = a$.

This statement says that 0 is an identity element for $*_3$. (When 0 is put in a pair with any number a, $*_3$ produces "identically" the same number a.)

Question: Can you give a counterexample to show that 0 is not an identity for $*_8$? Is there a number which is an identity element for $*_8$?

Let us look again at the operational system $(Z_3,+)$ studied in Chapter 1. The appropriate table is shown below:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The table shows clearly that + is an operation on the set Z_3. Why? Furthermore, we see that 0 is the identity element. Now, note the following assignments:

$$(0,0) \longrightarrow 0$$
$$(1,2) \longrightarrow 0$$
$$(2,1) \longrightarrow 0$$

What we have done is to list the ordered pairs of numbers which are assigned the identity element 0. As you recall, the numbers in such a pair are called additive inverses. The numbers 2 and 1 are inverses, since $2 + 1 = 0$. We also say that "2 is the inverse of 1" and "1 is the inverse of 2." This is the same way we shall use the word "inverse" when speaking of any operational system.

Question: What is the inverse of 0 in $(Z_3,+)$?

In this section, we have looked at four important features of operations: commutativity, associativity, identity element, and inverse elements. Let us now try to summarize them by using the "*" symbol to denote a binary operation on a set S.

1. Commutativity

   $*$ is commutative if for every $\underline{a}$ in S, and every $\underline{b}$ in S,

   $$a*b = b*a.$$

2. Associativity

   $*$ is associative if for every $\underline{a}$ in S, every $\underline{b}$ in S, and every $\underline{c}$ in S,

   $$(a*b)*c = a*(b*c).$$

3. Identity

   Suppose $\underline{e}$ is an element of the set S. $\underline{e}$ is an identity element for $(S,*)$ if for every $\underline{a}$ in S,

   $$a*e = e*a = a.$$

4. Inverse

   Suppose $\underline{e}$ is an identity element of $*$. Then $\underline{a}$ and $\underline{b}$ are inverses of each other if

   $$a*b = b*a = e.$$

In the exercises, you will have a chance to apply these definitions to many different operations. This should help you to see clearly what they mean.

2.10 Exercises

1. Tell what whole number is named by each of the following. Warning: Some of the expressions do not name any whole number at all.

(a) 82 + 517
(b) 517 + 82
(c) 517 - 82
(d) 82 - 517
(e) 82 x 517
(f) 517 x 82
(g) 816 ÷ 8
(h) 8 ÷ 816

2. Which of the following are true for every whole number a, and every whole number b?

(a) $a + b = b + a$
(b) $a - b = b - a$
(c) $a \cdot b = b \cdot a$
(d) $a \div b = b \div a$

3. Which of the following statements are true?

(a) Addition of whole numbers is commutative.
(b) Subtraction of whole numbers is commutative.
(c) Multiplication of whole numbers is commutative.
(d) Division of whole numbers is commutative.

4. (a) Are there any whole numbers a and b for which

$$a - b = b - a?$$

(b) Are there any whole numbers a and b for which

$$a \div b = b \div a?$$

5. Look again at the six operations defined in Section 2.5. Which of these are commutative operations? (Give a counterexample for each operation which is not commutative.)

6. Tell what whole number is named by each of the following:

(a) (12 + 6) + 2
(b) 12 + (6 + 2)
(c) 12 - (6 - 2)
(d) (12 - 6) - 2
(e) (12 x 6) x 2
(f) 12 x (6 x 2)
(g) (12 ÷ 6) ÷ 2
(h) 12 ÷ (6 ÷ 2)

7. Which of the following are true for every whole number a, every whole number b, and every whole number c?
   (a) $(a + b) + c = a + (b + c)$
   (b) $(a - b) - c = a - (b - c)$
   (c) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
   (d) $(a \div b) \div c = a \div (b \div c)$

8. Which of the following statements are true?
   (a) Addition of whole numbers is associative.
   (b) Subtraction of whole numbers is associative.
   (c) Multiplication of whole numbers is associative.
   (d) Division of whole numbers is associative.

9. (a) Are there any whole numbers a, b and c for which
   $$(a - b) - c = a - (b - c)\ ?$$
   (b) Are there any whole numbers a, b and c for which
   $$(a \div b) \div c = a \div (b \div c)?$$

10. Look again at the six operations defined in Section 2.5. Which of these do you think are associative operations? Try to find a counterexample for each operation which is not associative.

11. (a) Evaluate the following:
    $15 + 0$; $0 + 15$; $312 + 0$; $0 + 312$.
    (b) Name an identity element for addition of whole numbers. Is there more than one identity element?
    (c) Evaluate the following:
    15×1; 1×15; 312×1; 1×312.
    (d) Name an identity element for multiplication of whole numbers. Is there more than one identity element?

12. Give a counterexample showing that the number 1 is not an identity element for the operation $*_3$ on the set of whole numbers.

13. Notice that although 2 - 0 = 2, it is not true that 0 - 2 yields 2. Therefore, 0 is not an identity element for subtraction of whole numbers. (Look again at the definition of identity element if you do not see why this is the case.) Is there an identity element for division of whole numbers?

14. Construct a table for $(Z_6, +)$.
    (a) Is + a commutative operation here? (How does the table show this?
    (b) Is + an associative operation? (Is there a counterexample?)
    (c) Is there an identity element in $(Z_6, +)$?
    (d) List all pairs of numbers which are inverses for +.

15. Construct a table for $(Z_6, \cdot)$.
    (a) Is $\cdot$ commutative?
    (b) Is $\cdot$ associative?
    (c) Is there an identity element in $(Z_6, \cdot)$?
    (d) List all pairs of numbers which are inverses for $\cdot$.

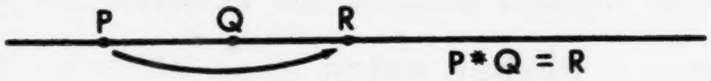
16. Look again at exercises 16 of Section 2.4, where we introduced an operation which assigned to every pair (P,Q) of points a midpoint. Call this operation mid in this problem.

(a) Is it true that PmidQ = QmidP for every point P and every point Q?

(b) Is it true that (PmidQ)midR = Pmid (QmidR) for every point P, every point Q, and every point R?

17. In this problem, we introduce a new operation on the set of pairs of points in a plane. (Think of a plane as simply a flat surface like the top of a desk.) Let P and Q be two points as below. Draw a line through these two points. We shall define P*Q for this problem as follows:

P*Q is the point R which is on the line through P and Q, on the "other side" of Q from P and at the same distance from Q as P.



We say in this case that "R is the reflection of P in Q."

(a) How could you define P*P?

(b) Show * is an operation on the set of points in the plane.

(c) Is this operation commutative?

(d) Is this operation associative?

(e) Does the operation have an identity element?

18. The table below defines an operation Δ over the set {a, b, c}.

Δ	b	a	c
b	b	a	c
a	a	c	b
c	c	b	a

(a) Is Δ an associative operation?

(b) Is Δ a commutative operation?

(c) Does Δ have an identity element?

(d) If there is an identity element, list all pairs of inverse elements.

19. (a) Consider the system (W,+); that is, addition of whole numbers.

Does the number 8 have an inverse in this system?

Does the number 0 have an inverse in this system?

(b) Consider the system (W,·); that is, multiplication of whole numbers.

Does the number 8 have an inverse in this system?

Is there a number which does have an inverse in this system?

2.11 Cancellation Laws

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$(Z_5,+)$

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$(Z_5,\cdot)$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$(Z_4,+)$

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$(Z_4,\cdot)$

Suppose that two people are asked to choose an element from Z_5 without telling what number they have chosen. Each, however, is to write some true sentence about his "unknown" number. The first person calls his number a and writes the following statement:

$$3 + a = 2.$$

The second person, without knowing what the first has written, calls his number b and writes the following:

$$3 + b = 2.$$

What conclusion can be drawn? It is apparent, from a glance at the $(Z_5,+)$ table, that a is 4 and that b is also 4, because 4 is the only number which, when added to 3, yields the number 2 in

$(Z_5,+)$. In order words, a and b name the same number, and we may write

$$a = b.$$

There is an important idea suggested here. Notice that from the statements made by the two people, we knew the following:

$$3 + a = 3 + b,$$

since the expressions on the two sides of the "=" sign were both given as equal to 2. We were then able to conclude:

$$a = b.$$

Would we have been able to draw the same conclusion if $3 + a$ and $3 + b$ had both been given as equal to 3, instead of 2? The answer is "yes," since in such a case both a and b would have to be 0. In fact, as you can verify yourself, as long as $3 + a = 3 + b$, we may conclude that a and b are the same number. Thus, we write:

In $(Z_5,+)$, if $3 + a = 3 + b$, then $a = b$.

There is nothing special about the number 3 in this argument. If, for instance, we know $2 + a = 2 + b$ or that $0 + a = 0 + b$ or that $1 + a = 1 + b$ or that $4 + a = 4 + b$, we can still conclude that a and b are the same number. To summarize, let a, b, c be numbers in Z_5.

If $c + a = c + b$ in $(Z_5,+)$, then $a = b$.

This is known as the cancellation law for addition in Z_5.

Now let us look at $(Z_4,\cdot)$. Suppose we know that a and b are two numbers in Z_4, and we know further that

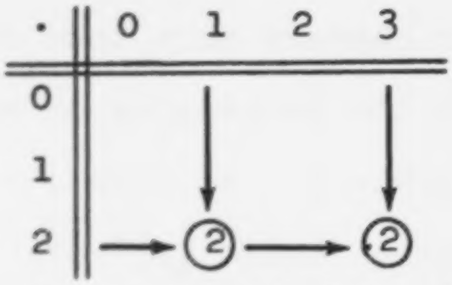
$$2 \cdot a = 2 \cdot b.$$

Can we conclude that a and b are the same number? Be careful!

At first, it might seem that this conclusion is justified. But look at the table for $(Z_4, \cdot)$. Do you see that

$$2 \cdot 1 = 2 \text{ and also } 2 \cdot 3 = 2?$$

This shows up clearly in the table since the number 2 appears more than once in a row:



In this case, $2 \cdot 1 = 2 \cdot 3$, but $1 \neq 3$. Hence, there is no cancellation law for multiplication in $(Z_4, \cdot)$. (Note also $2 \cdot 0 = 2 \cdot 2$, but $0 \neq 2$.)

Next, look at the table for $(Z_5, \cdot)$. Is there any number which appears more than once in any row of the table? Surely 0 does, since every entry in the first row is "0." So, even if we know

$$0 \cdot a = 0 \cdot b,$$

we cannot conclude that $a = b$. (For example, a might be 2, and b might be 3; yet $0 \cdot 2 = 0 \cdot 3$.) However, no number except 0 appears more than once in any row. Therefore,

> In $(Z_5, \cdot)$, if $c \neq 0$ and $c \cdot a = c \cdot b$,
> then $a = b$.

Thus we have a cancellation law in $(Z_5, \cdot)$ provided the numbers we "cancel" are not zeros.

Question: Examine the table for $(Z_4, +)$. Is there a cancellation law in this system? Is there an easy way to tell from the table?

In the following examples, we investigate some cancellation laws involving whole numbers. Specifically, we shall work with the systems (W,+) and (W,·).

Example 1: If $4 + a = 4 + b$, is it true that $a = b$?
The answer, of course, is "yes." Recall that in the addition table for whole numbers, no number appears more than once in any row (although the table goes on without end).

Example 2: If $a + 4 = b + 4$, does $a = b$?
The answer again is "yes." In fact, since + is a commutative operation on W, this is essentially the same as Example 1.

Example 3: If $4 \cdot a = 4 \cdot b$, does $a = b$?
Once again, the answer is "yes." For instance, if $4 \cdot a = 20$, and $4 \cdot b = 20$, then both a and b are 5. Because of commutativity, we can also say, "If $a \cdot 4 = b \cdot 4$, then $a = b$."

Example 4: If $0 \cdot a = 0 \cdot b$, does $a = b$?
NO! Recall that in the multiplication table for whole numbers, "0" is the entry in every cell of the first row. Thus $0 \cdot 2 = 0 \cdot 58$, since both products are 0; but $2 \neq 58$.

From examples such as these, it seems reasonable to formulate the following cancellation laws for addition and multiplication of whole numbers.

If a, b and c are whole numbers, and

if $c + a = c + b$, then $a = b$.

If a, b and c are whole numbers with $c \neq 0$, and

if $c \cdot a = c \cdot b$, then $a = b$.

Are you clear as to why we require $c \neq 0$ in the cancellation law for multiplication of whole numbers? (If not, see Example 4 above.)

Since addition and multiplication of whole numbers are each commutative operations, these cancellation laws could just as well have been stated in the following way

If $a + c = b + c$, then $a = b$.

If $a \cdot c = b \cdot c$ (and $c \neq 0$), then $a = b$.

We have now seen several systems in which cancellation laws are possible, and at least one, $(Z_4, \cdot)$, where there is no cancellation law. The notion of a cancellation law in an operational system may be defined in general as follows:

Definition: If $(S,*)$ is an operational system, we say that there is a cancellation law in $(S,*)$ provided that the following holds. If a, b, c are in S and

if $a * c = b * c$, then $a = b$.

2.12 Exercises

1. Suppose that a and b are whole numbers such that

$5 \cdot a = 95$ and $5 \cdot b = 95$

What number is a? What number is b?

2. Suppose that x and y are whole numbers such that

$x + 79 = 117$ and $y + 79 = 117$

What number is x? What number is y?

3. Suppose a, b, and c are whole numbers. What conclusions can you draw from the following?

   (a) $c + a = c + b$.

   (b) $c \cdot a = c \cdot b$, where $c \neq 0$.

   (c) $0 \cdot a = 0 \cdot b$.

4. Consider again the "maximizing" operation on the whole numbers.

   (a) Suppose there are two whole numbers a and b such that

   $$4 \max a = 4 \max b.$$

   Can you conclude that $a = b$?

   (b) Is there a cancellation law for (W,max)?

5. Let mid be the operation which assigns to every pair of points (P,Q) their midpoint. (See problem 16 of Section 2.4.) Is there a cancellation law for this operation; that is, if P mid Q = P mid S, where P,Q, and S are points, can you be sure that Q and S name the same point?

6. For which of the following systems are there cancellation laws?

   (a) $(Z_8,+)$ (b) $(Z_9,+)$ (c) $(Z_8,\cdot)$ (d) $(Z_9,\cdot)$

7. From which of the following statements can you conclude that $a = b$? (Parts (a) through (g) refer to whole numbers.)

   (a) $2 + a = 2 + b$

   (b) $0 + a = 0 + b$

   (c) $2 \cdot a = 2 \cdot b$

   (d) $0 \cdot a = 0 \cdot b$

   (e) $2 \max a = 2 \max b$

   (f) $2^a = 2^b$ (where a and b are not zero)

   (g) $a^2 = b^2$

(h) $2 + a = 2 + b$ in $(Z_3,+)$

(i) $2 \cdot a = 2 \cdot b$ in $(Z_3,\cdot)$

(j) $2 + a = 2 + b$ in $(Z_4,+)$

(k) $2 \cdot a = 2 \cdot b$ in $(Z_4,\cdot)$

8. Let * be the operation which assigns to any ordered pair of points (P,Q) in a plane the reflection of P in Q. (See exercise 17 of Section 2.10.) Is there a cancellation law for this operation?

9. The following table defines an operation on the set

$$\{a, b, c\}.$$

Is there a cancellation law for this operation?

*	a	b	c
a	a	b	c
b	b	c	b
c	c	a	b

10. Make up two new operations over the set W of whole numbers, so that one of the operations has a cancellation law and the other one does not.

11. The sum of two even whole numbers is an even number. We might abbreviate this statement as

even + even = even.

In the same way, we can state

odd + odd = even;

even + odd = odd;

odd + even = odd.

Now we consider the set S = {even, odd} having two elements. We can construct the following operational table:

+	even	odd
even	even	odd
odd	odd	even

(a) In (S,+), is + associative?

(b) Is + commutative?

(c) Is there an identity element?

(d) Does each element have an inverse?

(e) Does the system (S,+) have a cancellation law?

12. Using the set $S = \{\text{even, odd}\}$ from problem 11, construct an operational table for the system $(S,\cdot)$ suggested by multiplication of odd and even integers.

(a) In $(S,\cdot)$, is $\cdot$ associative?

(b) Is $\cdot$ commutative?

(c) Is there an identity element?

(d) Does each element have an inverse?

(e) Does the system $(S,\cdot)$ have a cancellation law?

2.13 Two operational Systems

Let S be the set

$$\{6, 8, 2, 4\},$$

a subset of the set of even whole numbers. We are going to introduce an operation on this set which we shall denote by the symbol "$\odot$" since it is closely related to multiplication of whole numbers. To begin with an illustration, consider the problem of making an assignment to the ordered pair (8,4). The product of 8 and 4 is 32. We shall keep only the last digit, and write

$$8 \odot 4 = 2.$$

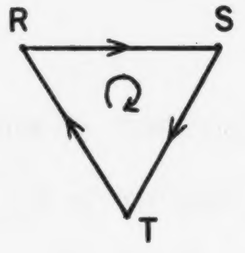
As another example, the ordered pair (2,8) shall be assigned the number 6 by the $\odot$ operation, since the ordinary product of 2 and 8 is 16, and the last digit of the numeral "16" is "6."

As you can see, the $\odot$ operation makes its assignments on the basis of certain digits; so we can call it digital multiplication. Printed below is an operational table showing the assignments for all ordered pairs of elements of S under digital multiplication.

$\odot$	6	8	2	4
6	6	8	2	4
8	8	4	6	2
2	2	6	4	8
4	4	2	8	6

From the table, you can see that we were justified in calling $\odot$ an operation on S. In the exercises that follow, you will be asked to investigate some of the properties of the operational system $(S, \odot)$.

For the second operational system, we use the set P of all points in a plane. For example, R and S are two points in the plane of this sheet of paper. If we are to have a binary operation on this set P of points, we must be able to assign to every ordered pair of points such as (R,S) some particular point of the plane. Let us agree to make the assignment in the following way:



(R,S)→T

Move your pencil from R to S (that is, from the first point of the ordered pair to the second). Then move your pencil (in a clockwise sense) to a point T so that R, S, and the point T are corners (or vertices) of an equilateral triangle. This point T is the one we shall assign to the pair (R,S).

What assignment could we make to an ordered pair such as (R,R)? If in such a case we agree simply to assign the point R itself, then we are able to make an assignment -- and only one assignment -- to every ordered pair of points. We now have an operation. Since a triangle helped us to define this operation, let us call the operation "tri." Thus, for the points above, we have

$$R \text{ tri } S = T.$$

Is S tri R the same as R tri S? In the exercises that follow, you will have a chance to answer questions such as this about the system (P,tri).

2.14 Exercises

Questions 1-6 are about the operational system $(S, \odot)$ explained in the text.

1. (a) Is $\odot$ commutative? If not, give a counterexample.
   (b) How does the pattern of the operational table show that your answer in (a) is correct?
2. (a) Compute $8 \odot (6 \odot 4)$ and $(8 \odot 6) \odot 4$.

(b) Is $\odot$ associative? Is there any way you can tell without testing every possible case?

3. (a) Is there an identity element in $(S, \odot)$?

   (b) Is there more than one identity element in $(S, \odot)$?

4. (a) What is the inverse element of 8 in $(S, \odot)$?

   (b) What number is its own inverse in $(S, \odot)$? Is there another?

5. (a) If $2 \odot a = 2 \odot b$, what can you conclude about a and b?

   (b) Is there a cancellation law in $(S, \odot)$? How can this question be answered by inspecting the table?

6. Solve the following open sentences in $(S, \odot)$:

   (a) $x \odot 2 = 6$
   (b) $2 \odot x = 2$
   (c) $x \odot x = 6$
   (d) $x \odot x = 8$
   (e) $(x \odot x) \odot x = 2$
   (f) $x \odot (x \odot x) = 2$

Questions 7-11 refer to the operational system (P,tri) discussed in the text.

7. Is tri commutative? If not, give a counterexample.

8. Is tri associative? (Try at least two different cases.)

9. Is there an identity element for (P,tri)? Is there more than one identity element?

10. Does every point have an inverse in (P,tri)? Defend your answer.

11. Is there a cancellation law in (P,tri)?

12. (a) Does the system $(S, \odot)$ have any properties which (P,tri) does not have?

    (b) Does the system (P,tri) have any properties which $(S, \odot)$ does not have?

2.15 What is a Group?

In this chapter we have studied many different operational systems and we have called attention to such properties as associativity, identity elements, and inverse elements. Because operational systems which possess these three properties play an important role in mathematics we give the special name group to any such system. That is, if (S,*) is an operational system such that

(1) * is associative;

(2) there is an identity element; and

(3) each element has an inverse,

then (S,*) is said to be a group.

Questions: Is $(Z_3, +)$ a group?

Is $(W,+)$ a group?

Notice that the operation in a group does not have to be commutative. However, it may be, and if it is, the group is called a commutative group.

Questions: Is $(Z_3,+)$ a commutative group?

Is $(W,+)$ a commutative group?

2.16 Exercises

Decide which of the following are commutative groups. Remember that there are four necessary properties, and each must be verified.

1. $(Z_4,+)$ 2. $(Z_4,\cdot)$ 3. (W,max)

4. $(S,\cdot)$, where $S = \{6, 8, 2, 4\}$ and $\cdot$ is digital multipli-

cation. (See Section 2.13.)

5. (P,tri) (See Section 2.13.)

2.17 Summary

1. An operation on a set S is an assignment of one and only one element of S to every ordered pair of elements of S. If an operation assigns c to the ordered pair (a,b), we may show the assignment as

   $$(a,b) \longrightarrow c.$$

   If a symbol such as "*" is used to identify the operation, the assignment may be shown as

   $$a * b = c.$$

   When * is an operation on set S, we denote the operational system by the pair (S,*).

2. If a and b are elements of S, and (S,*) is an operational system, then a sentence such as

   $$a * x = b$$

   is an open sentence in the system. Any element of S which, when substituted for x, gives a true statement, is called a solution of the open sentence.

3. There are certain properties of operations which are important. For example, if (S,*) is an operational system and we let a, b and c represent elements in S, then

   * is commutative if $a * b = b * a$ for every a and b;

   * is associative if $(a * b) * c = a * (b * c)$ for every a, b and c,

e is an identity element of (S,*) if $e * a =$
$a * e = a$ for every a;

a and b are inverse elements in (S,*) if
$a * b = b * a = e$ where e is an identity element.

If in this system $a * c = b * c$ always implies $a = b$, we say that (S,*) has a cancellation law.

4. (W,+) and (W,·) are two important systems involving the whole numbers. These operational systems have the following properties:

In (W,+), + is associative;
+ is commutative;
there is an identity element, 0;
there is a cancellation law.

In (W,·), · is associative;
· is commutative;
there is an identity element, 1;
if $c \neq 0$, $a \cdot c = b \cdot c$ implies $a = b$.

2.18 Review Exercises

1. Tell what number is assigned to the ordered pair (7,2) in each of the following systems:

(a) (W,+)
(b) (W,·)
(c) (W,max)
(d) $(Z_{12},+)$
(e) $(Z_{12},\cdot)$

2. List all pairs which are assigned the number 4 in each of the following systems:

(a) (W,+)
(b) (W,·)
(c) (W,max)
(d) $(Z_{12},+)$
(e) $(Z_{12},\cdot)$

3. Tell what whole number (if any) is named by each of the following:

(a) $867 + 245$	(h) $5 \cdot 87$
(b) $245 + 867$	(i) $87 \div 5$
(c) $867 - 245$	(j) $5 \div 87$
(d) $245 - 867$	(k) 3^2
(e) 867 max 245	(l) 3^4
(f) 245 max 867	(m) 4^3
(g) $87 \cdot 5$	

4. Which of the following are operations on the set W of whole numbers?
   - (a) addition
   - (b) multiplication
   - (c) subtraction
   - (d) division
   - (e) maximizing
   - (f) raising to a power

5. Which of the following statements are true for every whole number a, every whole number b, and every whole number c?
   - (a) $a + b = b + a$
   - (b) $a \cdot b = b \cdot a$
   - (c) $a - b = b - a$
   - (d) $a \div b = b \div a$
   - (e) a max b = b max a
   - (f) $a^b = b^a$

6. Find the whole number named by each of the following, if a is 12, b is 6, and c is 2.

   (a) $(a + b) + c$
   (b) $a + (b + c)$
   (c) $(a - b) - c$
   (d) $a - (b - c)$
   (e) $(a \cdot b) \cdot c$
   (f) $a \cdot (b \cdot c)$
   (g) $(a \div b) \div c$
   (h) $a \div (b \div c)$
   (i) $(a \text{ max } b) \text{ max } c$
   (j) $a \text{ max } (b \text{ max } c)$

7. Find the number named by each of the following if a is 4, b is 2, and c is 3.

   (a) $(a^b)^c$
   (b) $a^{(b^c)}$

8. Which of the following are associative?

   (a) addition of whole numbers
   (b) division of whole numbers
   (c) subtraction of whole numbers
   (d) multiplication of whole numbers
   (e) maximizing with whole numbers
   (f) raising to a power with natural numbers

9. "Averaging" is not an operation on the whole numbers, but assignments can be made to certain pairs. Let "aVb" mean "the average of a and b."

   (a) What is 8 V (12 V 20)?
   (b) What is (8 V 12) V 20?
   (c) Is averaging associative?

10. Find what number each of the following names in W.

   (a) $((6 + 7) \cdot 3) + 16$
   (b) $((9 \cdot 5) \text{ max } 46) + 156$
   (c) $100 \cdot ((2^6) + 17)$
   (d) $((5 + 7) \cdot (3 + 17)) \cdot 10$
   (e) $((5 \text{ max } 7) \cdot 8) + ((5^2) + 3)$

11. Find all whole number solutions of the following open sentences. If there is no whole number solution, say so.

(a) $156 + x = 217$
(b) $89 + a = 89$
(c) $89 + a = 88$
(d) $a \cdot 14 = 98$
(e) $a \cdot 14 = 99$
(f) $14 \cdot a = 14$
(g) $14 \cdot a = 0$
(h) $4 \max n = 4$
(i) $4 \max n = 5$
(j) $4 \max n = 3$
(k) $a^3 = 8$
(l) $3^a = 8$
(m) $1^a = 1$
(n) $1^a = 2$
(o) $a + a = 100$
(p) $a \cdot a = 100$
(q) $a^a = 100$
(r) $n^n = 27$
(s) $a \max a = 100$
(t) $(2 \max a)^2 = 4$

12. Find what whole number is named by each of the following if $a = 2$ and $b = 5$. (See Section 2.6, Exercise 23.)

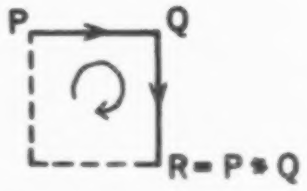
(a) $a^3 + 2$
(b) $2a^3$
(c) $(2a)^3$
(d) $(a + b)^2$
(e) $a^2 + b^2$
(f) $a^2 + [\ 2 \cdot (a \cdot b)] + b^2$
(g) $a + b^2$
(h) $2a^3 + 5$
(i) $2 \cdot [(a^3) + 5]$
(j) $(a \max b)^2$
(k) $a \max (b^2)$

13. If each of the following is taken to be a true statement about the whole numbers a and b, from which can we conclude that $a = b$?

(a) $5 + a = 5 + b$
(b) $0 + a = 0 + b$
(c) $5 \cdot a = 5 \cdot b$
(d) $0 \cdot a = 0 \cdot b$
(e) $3 \max a = 3 \max b$
(f) $a^3 = b^3$

14. Consider all ordered pairs of points in a plane. If (P,Q) is an ordered pair of points, let $P*Q$ be found in the following way:

Take P and Q as corners of a square, and let R be the third corner of the square if you move in a "clockwise" way from P to Q to R. (See diagram at right.) Then $P*Q = R$. Answer the following questions:



(a) What point can be assigned to a pair such as (Q,Q)?

(b) Is * an operation on the set of points in a plane?

(c) Is * commutative?

(d) Is * associative?

(e) Is there an identity element?

(f) Is there a cancellation law?

15. Take the set $S = \{0, 1\}$, and construct tables for all possible binary operations on S.

CHAPTER 3

MATHEMATICAL MAPPINGS

3.1 Assignments and Mappings

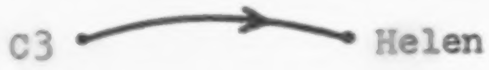
In Chapter 2, assignments to the ordered pairs of elements of a set were studied. For example, the operation of addition of whole numbers assigns to each ordered pair of whole numbers a unique whole number, called the sum. Thus, $(18,9) \longrightarrow 27$ by the operation of addition.

In this chapter we shall examine other kinds of assignments and, in particular, the special kind of assignment that is called a mapping. Now let us look at some examples of assignments, some of which are mappings and some of which are not.

Example 1. There are 5 tables in a home economics room in in a school labelled C1, C2, C3, C4 and C5. The chart below gives the assignment of girls in a home economics class to the tables.

Table	Name
C1	--
C2	Judy
C2	Mary
C3	Helen
C4	Louise
C4	Sandra
C5	Janice
C5	Carol

A convenient way to represent this assignment is to construct an arrow diagram for the assignment. To show that Helen is assigned to table C3 by this assignment, we draw an arrow from "C3" to "Helen" as shown.



We say that C3 is at the origin of the arrow and that Helen is at the terminus of the arrow. Then, listing the two sets given, we construct the arrow diagram showing all the assignments.

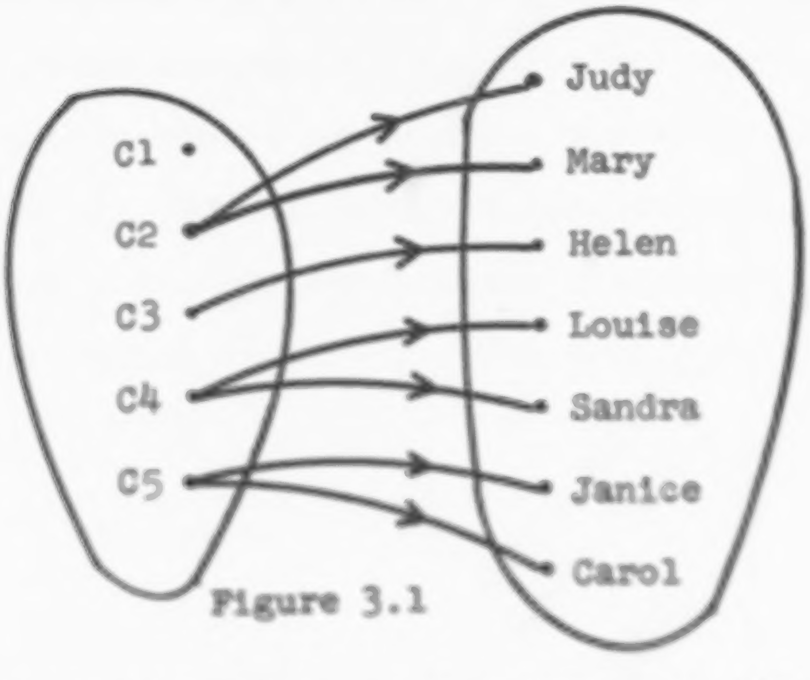

C1
C2
C3
C4
C5
Judy
Mary
Helen
Louise
Sandra
Janice
Carol

Figure 3.1

The diagram shows, for instance, that both Louise and Sandra are assigned to table C4.

Example 2. A basketball program lists the heights of the boys in the first team as follows:

Name	Height in inches
John Hammond	73
Al Parks	77
Bert Moyer	70
Fred Clark	73
Steve Hanson	68

By this chart a whole number is assigned to each boy.

Figure 3.2 shows an arrow diagram for this assignment.

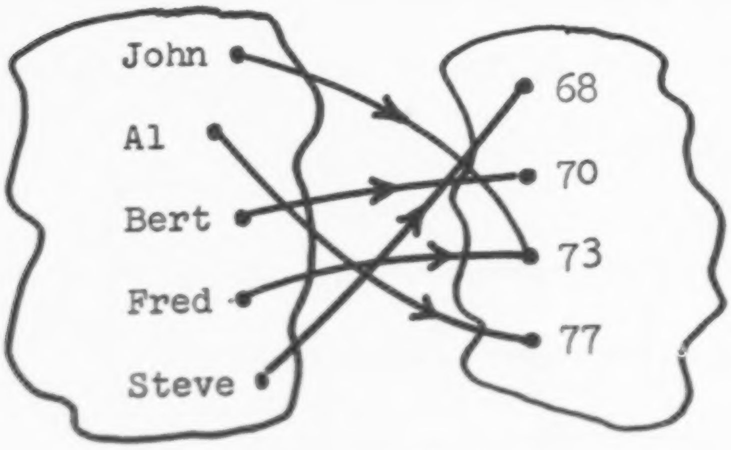


Figure 3.2

Example 3. Assign to each whole number in the set of whole numbers {2, 3, 4, 5, 6, 7} each whole number in the set {1, 2, 3, 4, 5, 6, 7} which it divides exactly. The arrow diagram for this assignment is shown in Figure 3.3.

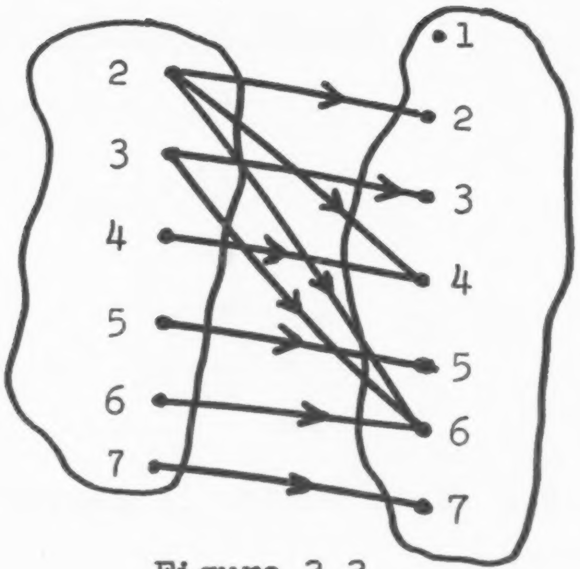

2
3
4
5
6
7
1
2
3
4
5
6
7

Figure 3.3

The diagram shows, for instance, that 3 divides 3 and 3 divides 6.

Example 4. Assign to each state of the United States its capital city.

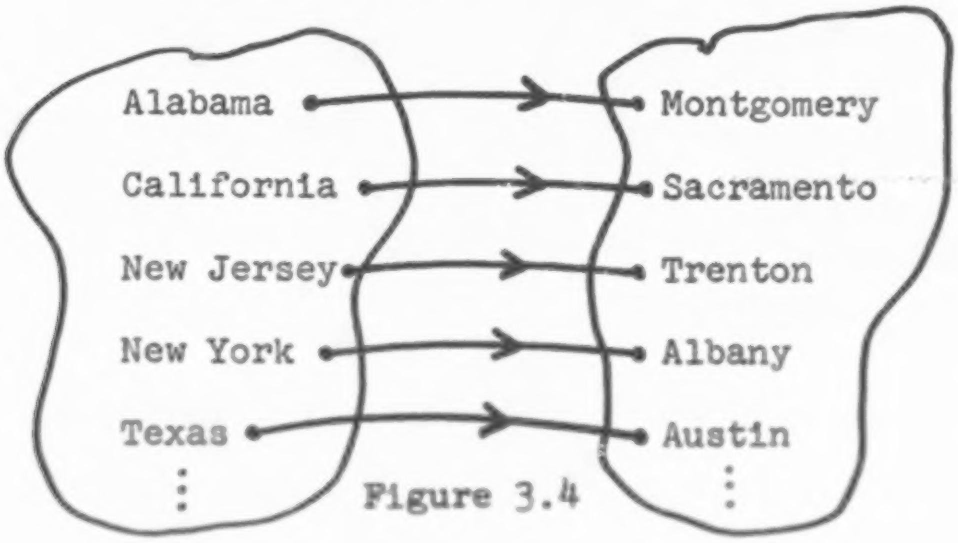

Alabama
California
New Jersey
New York
Texas
⋮
Montgomery
Sacramento
Trenton
Albany
Austin
⋮

Figure 3.4

In this case the arrow diagram does not give the complete assignment but the complete assignment could be given, perhaps with the aid of an atlas.

Example 5. Assign to each whole number a whole number that is 5 more than the given whole number.

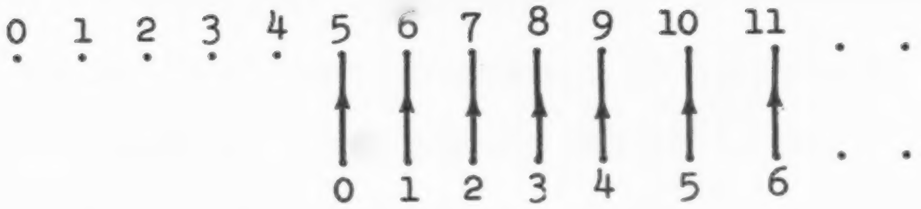

0 1 2 3 4 5 6 7 8 9 10 11 . .
0 1 2 3 4 5 6 . .

Figure 3.5

In this arrow diagram only part of the assignment is given. It is impossible to give the complete assignment by an arrow diagram because the set of whole numbers could never be completely listed in this way.

Example 6. Consider the set of children {Mary, Steve, Joe, Janet, Peter, and Harry}, and assign to each child his father. The complete assignment is given by an arrow diagram in Figure 3.6.

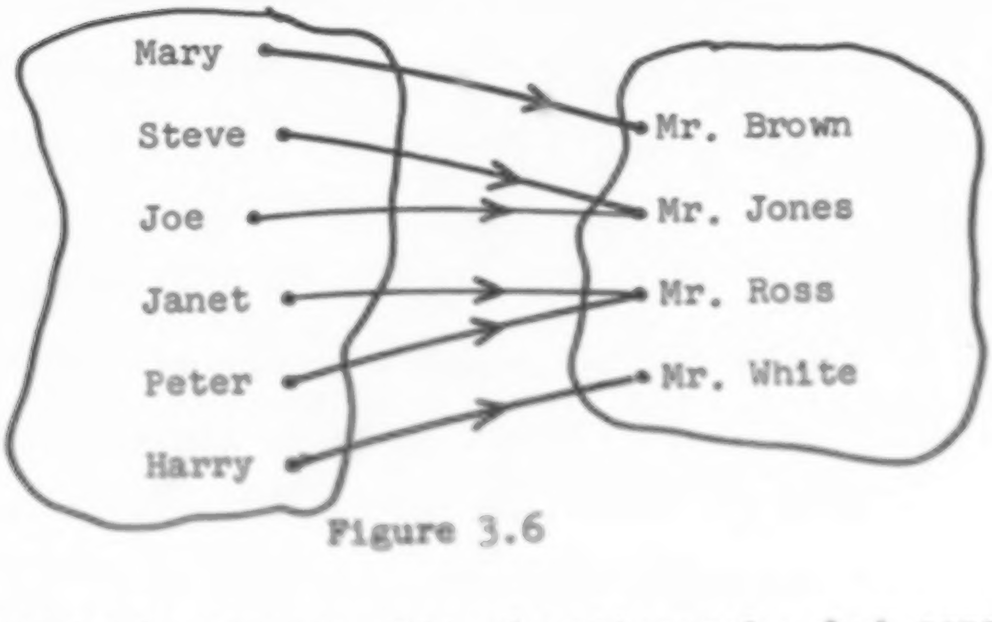

Mary
Steve
Joe
Janet
Peter
Harry
Mr. Brown
Mr. Jones
Mr. Ross
Mr. White

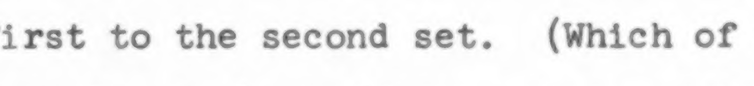
Figure 3.6

We could list many examples of assignments, but some things should be clear to you from the preceding examples. In each example there is a first set and a second set. Sometimes each element of the first set is assigned an element of the second set, and sometimes not. In Example 1, the table C1 is assigned no student. Sometimes an element of the first set is assigned more than one element of the second set. In Example 3, 2 is assigned 2, 4, and 6. These assignments are *not* mappings.

Now let us focus our attention on Example 4. In this example note that to each element in the first set is assigned *at least one* element of the second set. Furthermore, to each element of the first set is assigned *only one* element of the second set. We say then that to each element of the first set there is assigned *exactly one* element of the second set. Assignments having this property are of great importance, both in mathematics and in its applications. Such an assignment is called a mapping of the first to the second set. (Which of the assignments given in Examples 2, 4, and 5 are also mappings?)

More formally, given two sets A and B, to have a mapping of A to B, to each element of A there must be assigned exactly one element of B. The method of assignment is often called a *rule of assignment*, or simply a *rule* for the mapping.

The first set, A, in a mapping is called the *domain* of the mapping. In Example 6, the domain A is the set of children {Mary, Steve, Joe, Janet, Peter, Harry}. Since, in this mapping, Steve is assigned Mr. Jones, we say that Mr. Jones is the *image*, mathematically speaking, of Steve.



We see that Steve, the member of the domain, appears at the origin of the arrow and that Mr. Jones, the image, appears at the terminus of the arrow.

The second set, B, in a mapping is often called the codomain. For Example 6, the codomain B = {Mr. Brown, Mr. Jones, Mr. Ross, Mr. White}. In some mappings each element of the second set is an image, as is the case in Example 6, but this is not the case for the assignment in Example 5, which is also a mapping. Notice that the set of images in a mapping, which is called the range of the mapping, may be either all of the second set, or a part of it.

In this section we have shown you several examples of assignments, and have begun the study of those special assignments which are called mappings. Later, we will study other mappings, particularly those in which the sets A and B are sets of numbers.

3.2 Exercises

1. Answer the following questions for each of the assignments in Examples 1-6 given in section 3.1.
   (a) Is each element of the first set assigned at least one element of the second set? If not, which elements are not assigned?
   (b) Is any element assigned more than one element of the second set? If so, which ones, and what are the elements of the second set assigned? (You may answer this question by using arrows.)
   (c) Is the assignment a mapping of the first set to the second set?

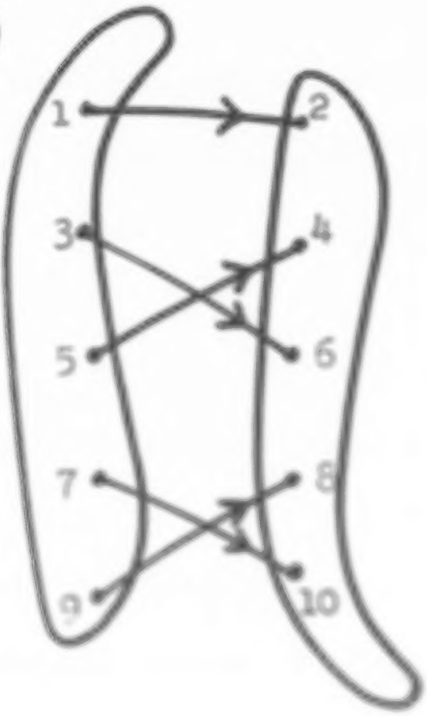
(d) For each assignment that is a mapping

  (1) list the domain and range;

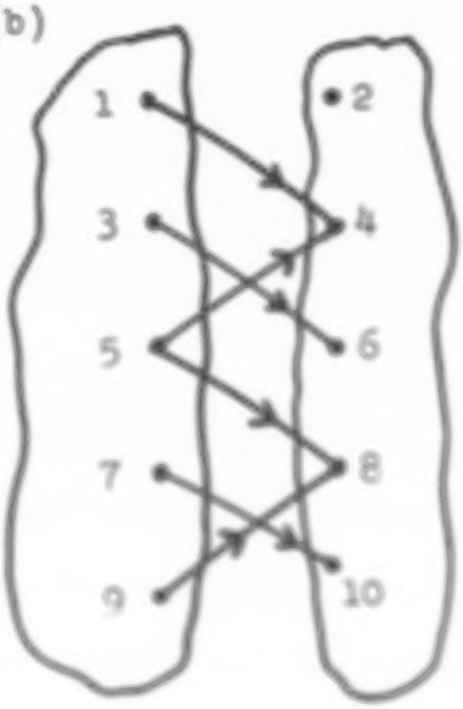
  (2) state whether or not the range is all of the second set.

2. Set A = [1, 3, 5, 7, 9] and B = [2, 4, 6, 8, 10]. Which of the following arrow diagrams represent mappings of A to B? In each case explain why the arrow diagram does or does not represent a mapping of A to B.

(a)


1
3
5
7
9
2
4
6
8
10

(b)


1
3
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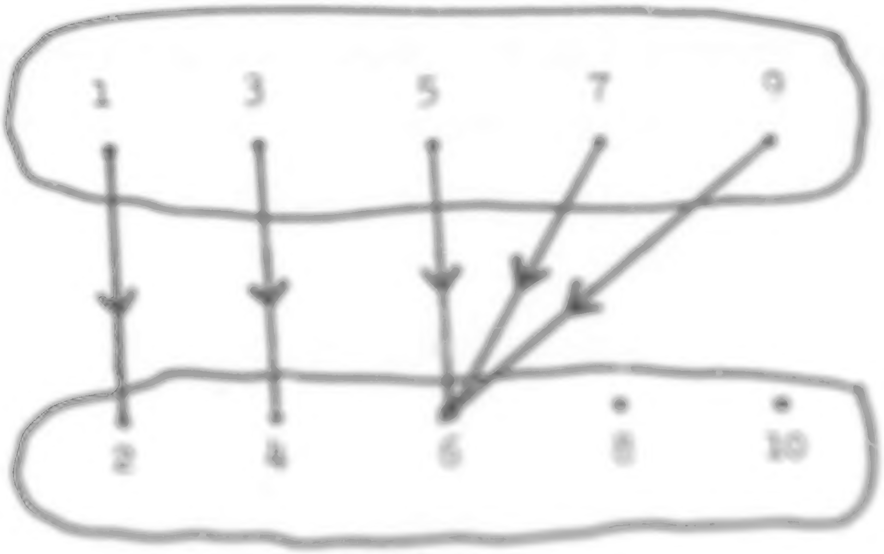
(c)


1
3
5
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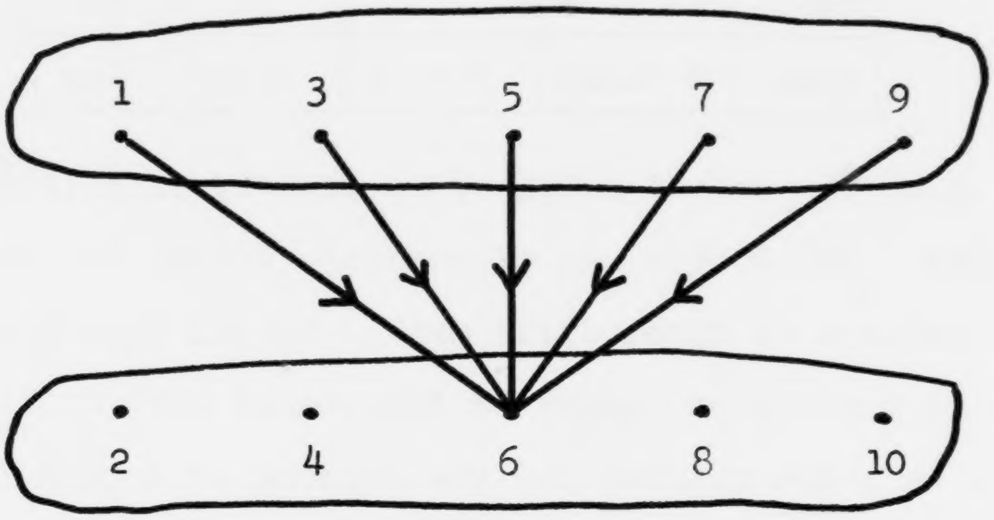
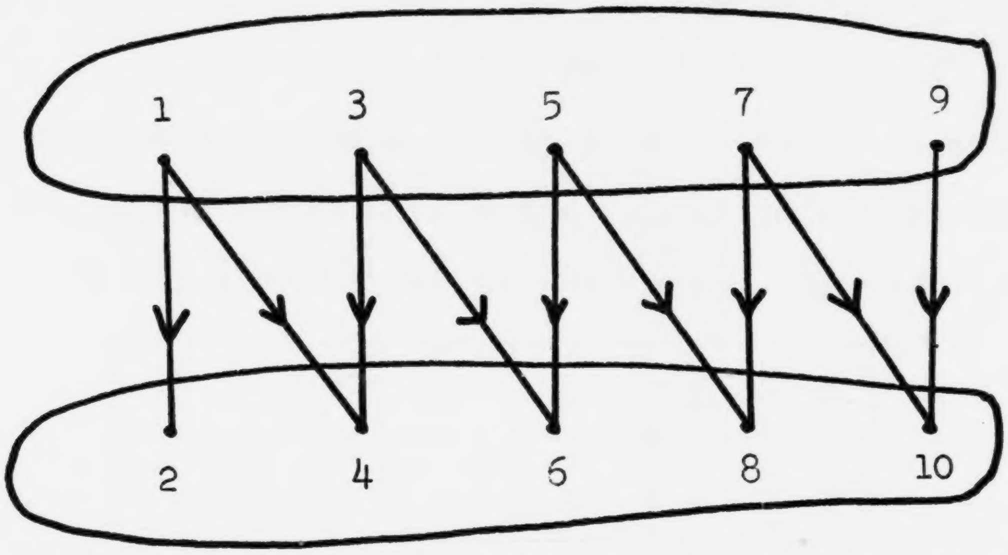
8
10

(d)



(e)



3. The following charts give the assignments to tables C1, C2, C3, C4, and C5 in the home economics room of the girls in three different classes. For each assignment

(a) draw an arrow diagram;

(b) state whether or not the assignment is a mapping;

(c) give a reason for your answer to (b).

1.

Table	C1	C2	C2	C3	C4	C5
Name	Jane	Elaine	Karen	Martha	Peggy	Alison

2.

Table	C1	C2	C3	C4	C5
Name	Noreen	Betty		Theresa	Eileen

3.

Table	C1	C2	C3	C4	C5
Name	Dolores	Cheryl	Betsy	Ann	Veronica

4. Let A be the set of weights, in ounces, of 5 letters to be mailed. Let B be a set of possible costs, in cents, of mailing letters by first class mail. Recall that post offices charge 6 cents per ounce or fractional part of an ounce. Draw an arrow diagram for the mapping of A to B if $A = \{3, 4\frac{1}{2}, 6\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$.

5. Let $A = \{1, 2, 3, 4, 5\}$ and let $B = A$.

   (a) Draw an arrow diagram of the mapping of A to B given by the following table, where each member of A has assigned as image the corresponding table entry for B.

A	1	2	3	4	5
B	4	2	1	2	4

   (b) What is the image of 3 in this mapping?

   (c) What is the range of this mapping?

   (d) Is the range the same as set B? Why?

   (e) Is any element of B the image of more than one element of A?

3.3 Mappings of Sets of Whole Numbers

In Section 3.1 mappings of A to B were considered for which the set A or the set B was not a set of numbers. For instance, in Example 2, the domain A is a set of boys, and the codomain B is a set of whole numbers. There are many other mappings with both the sets A and B sets of numbers. We shall now consider mappings of A to B in which the sets A and B are sets of whole numbers.

Example 1. Let the domain A = {2, 3, 12, 7, 4}, and let the codomain B = {6, 9, 12, 21, 36}. Let the method of assigning images be: to find the image of a number in A, multiply the number by 3. An arrow diagram for this mapping could then be given as is shown below. (In this example and hereafter we shall write 3n to mean 3 · n; similarly, $\frac{1}{2}$n means $\frac{1}{2}$ · n, 7x means 7 · x, etc.)

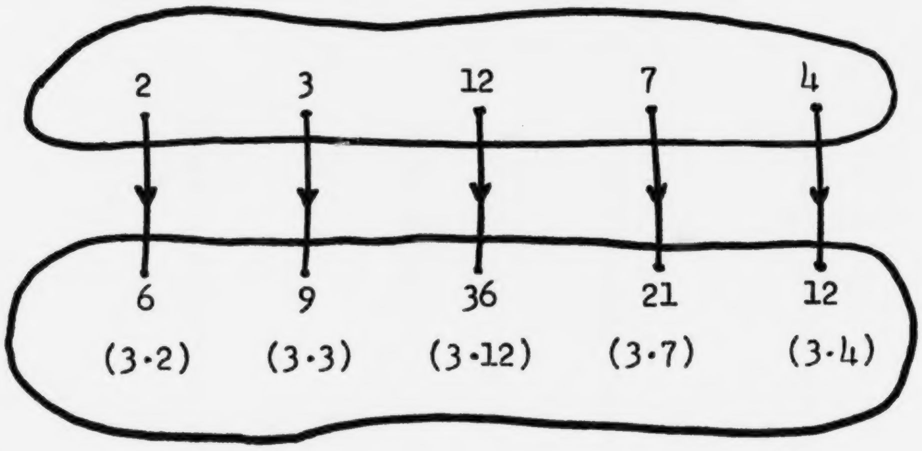


Figure 3.7

It is convenient to designate the method of assigning images in this mapping as n ⟶ 3n where n is any number in the domain A of the mapping. We can read "n ⟶ 3n" in any one of the following ways:

(1) The image of n is 3n.

(2) n is mapped onto 3n.

(3) To n is assigned 3n.

We refer to "n ⟶ 3n" as the rule for the mapping.

Example 2. Let A be the set of whole numbers, W, and let B also be the set of whole numbers. Let the rule of the mapping be: to each whole number in A is assigned 3 times that whole number. Since the domain of this mapping is infinite, we can only give an incomplete arrow diagram as shown in Figure 3.8, where we agree that the assignments continue in the same way for the remaining elements of the domain.

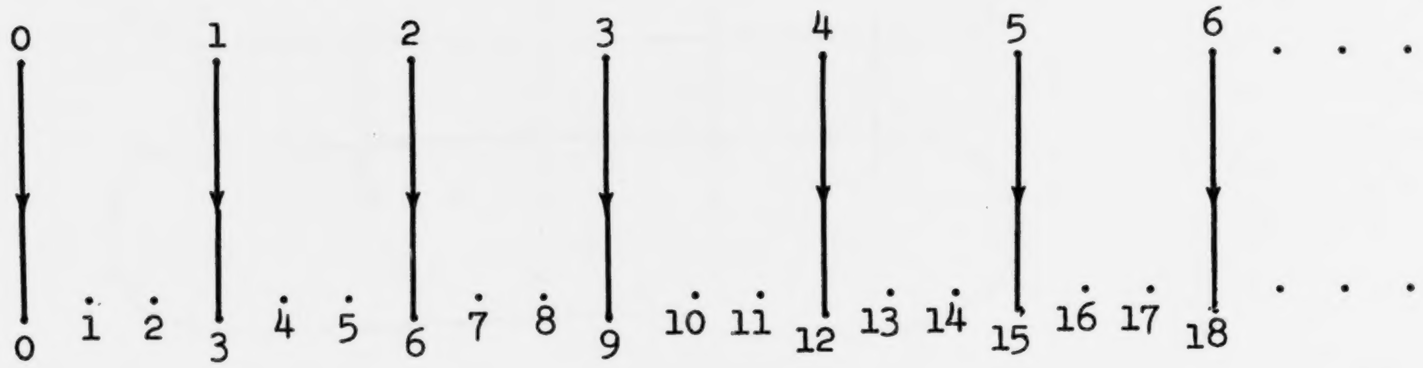


Figure 3.8

Note that for this mapping the domain is W and the range is {0, 3, 6, 9, ...}; that is, the set of multiples of 3. It is clear that even though the mappings in Examples 1 and 2 have the same rule, n ⟶ 3n, they are not the same mapping; one is finite, the other is infinite. In what other way do they differ?

Example 3. Let $A = \{6, 8, 10, \ldots\}$ and let $B = W$. Let the rule of assignment be $n \longrightarrow (\frac{1}{2}n) - 3$. First, let us find the images of some of the numbers in A.

The image of 6: $6 \longrightarrow (\frac{1}{2} \cdot 6) - 3 = 3 - 3 = 0$

The image of 8: $8 \longrightarrow (\frac{1}{2} \cdot 8) - 3 = 4 - 3 = 1$

The image of 10: $10 \longrightarrow (\frac{1}{2} \cdot 10) - 3 = 5 - 3 = 2$

Thus, an incomplete arrow diagram would be:

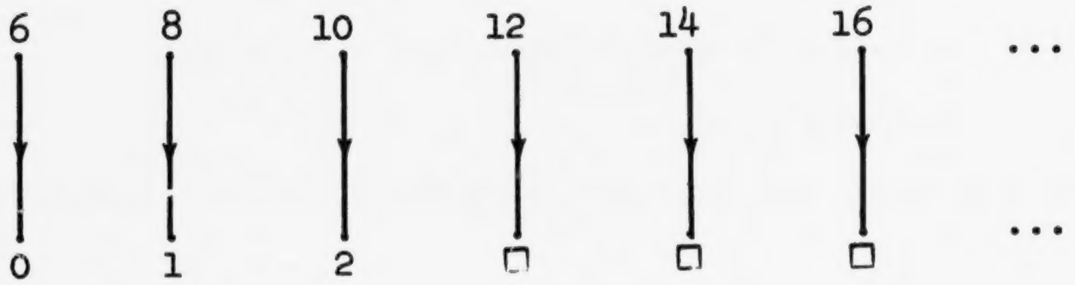


Figure 3.9

You should be able to supply the missing images in this diagram.

Now suppose we construct an assignment of W to W using the rule $n \longrightarrow (\frac{1}{2}n) - 3$. Then this assignment is not a mapping, for there are many whole numbers that are assigned no image by this rule. Consider 7.

$$7 \longrightarrow (\frac{1}{2} \cdot 7) - 3 = 3\frac{1}{2} - 3 = \frac{1}{2}.$$

$\frac{1}{2}$ is not a whole number. Is any odd number assigned a whole number by this rule? Now, consider 4.

$4 \longrightarrow (\frac{1}{2} \cdot 4) - 3 = 2 - 3.$

At present, we can give no number for the difference 2 - 3. Is any whole number less than 6 assigned a whole number by this rule?

From these examples we see that giving a rule of assignment is not enough to define a mapping completely. We must also be given two sets A and B so that the given rule

(1) assigns to each element of A at least one element of B; and

(2) assigns to each element of A only one element of B.

Then we can say that the rule of assignment defines a mapping of A to B.

We now consider another kind of arrow diagram.

Example 4.

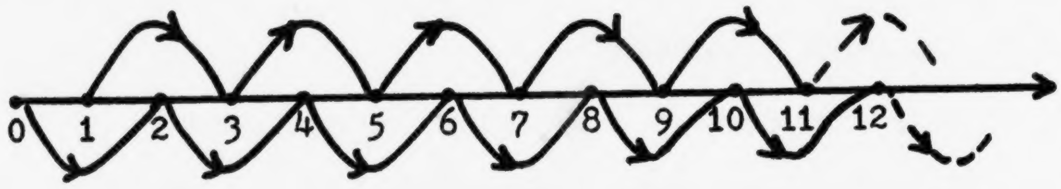


Figure 3.10

Even though the last number shown on this number line is 12, you are expected to assume that we are talking here about the set of all whole numbers, and that the arrows continue in the same pattern. Look at the arrow starting at the point labelled 0. Where does it end? This arrow shows that the image of 0 is 2; that is, $0 \longrightarrow 2$. For the numbers shown in the diagram, the rule $n \longrightarrow n + 2$ is satisfactory. Let us use this rule for all of W. Does each whole number, shown or not in the diagram, have an image? Exactly one image? If we interpret each arrow as connecting two whole numbers we have an example of an arrow diagram of a mapping of W to W on a line. Using the diagram, find the image of 3, of 4, of 7. Is there a number whose image is 3? 4? 7?

Since every whole number is at the origin of an arrow, the domain of the mapping is W. 0 and 1 are not at the terminus of any arrow so the range of this mapping is {2, 3, 4, 5, 6, ...}, that is, all whole numbers greater than 1.

Example 5. Let A = {0, 1, 2, 3, 4, 5}, B = W, and let the rule of assignment be $n \longrightarrow 5 - n$. We then obtain the following arrow diagram on a line.

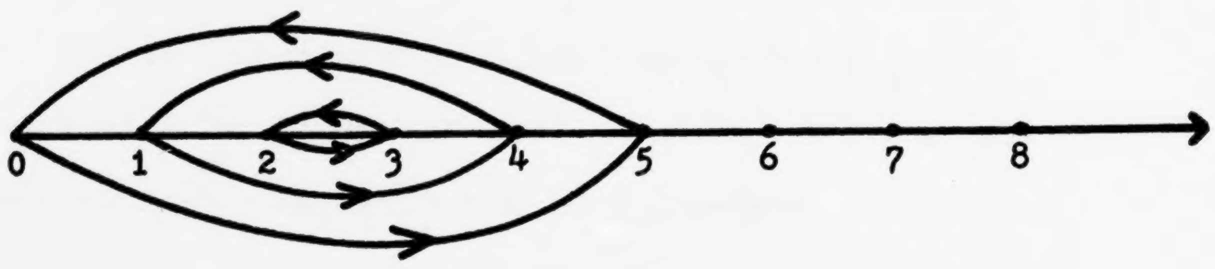


Figure 3.11

You should check to be sure that it is indeed a mapping of A to W. Every member of the set A is at the origin of an arrow and also at the terminus of an arrow. Hence, the domain and the range are the same set, A.

Example 6. Let M = {0, 1, 2, 3, 4, 5, 6, 7} and consider the mapping of M to the whole numbers W given by the following arrow diagram on a line.

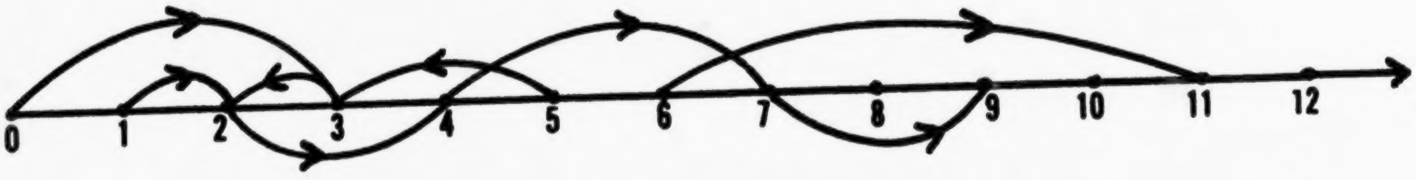


Figure 3.12

You can easily check that this diagram represents a mapping by noting that each number in M is at the origin of exactly one arrow of the diagram. In this case there is no easily seen rule of the form n ⟶ ? for this mapping. However, the diagram itself serves quite nicely as a rule. Thus we find the image of 3 is 2, since 2 ⟵ 3 on the diagram. What is the image of 2? of 5? of 6? What is the range of this mapping?

Example 7. Let $S = \{0, 1, 2, 3, 4\}$, and consider the mapping of S to W given by the following arrow diagram.

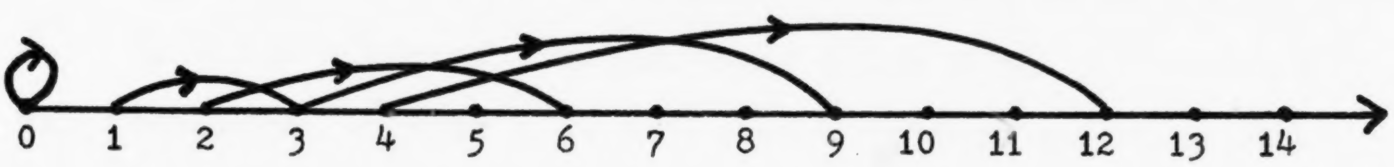


Figure 3.13

It is easy to see that there is a rule of the form $n \longrightarrow \square$ for this mapping, it is $n \longrightarrow 3n$. For instance, $0 \longrightarrow 3 \cdot 0 = 0$, $1 \longrightarrow 3 \cdot 1 = 3$, and $2 \longrightarrow 3 \cdot 2 = 6$.

Looking at our examples we see that in an arrow diagram of a mapping, each element of the domain appears at the origin of an arrow. Because of this we agree from now on that whenever we are given an arrow diagram for a mapping, the domain is understood to be the set of elements which appear at the origins of the arrows. The codomain, unless otherwise given, will be W, the set of whole numbers.

3.4 Exercises

1. Take W for the first set in a mapping and $N = \{1, 2, 3, \ldots\}$ for the second set. Let $n \longrightarrow n + 3$ be the rule for the mapping.

   (a) What is the image of 0? of 38? of 1359?

(b) Make an incomplete arrow diagram on a line for this mapping, showing the image of each whole number less than 13.

2. Try to repeat exercise 1 using sets W and N and the rule $n \longrightarrow n - 2$. Do 0 and 1 have images? Choose a set A of whole numbers so that $n \longrightarrow n - 2$ is a rule of assignment for a mapping of A to N. (More than one answer is possible.)

3. Make an incomplete arrow diagram on a line for the mapping of W to W having the rule $n \longrightarrow (2n) + 1$. Show the image of each whole number less than 13 on your diagram.

4. In this exercise you are asked to map $A = \{3, 4, 5\}$ into the set of whole numbers for each of the rules given below. Tell whether the statement accompanying each rule is true or false.

   (a) $n \longrightarrow 2n$. The image of 4 is between the image of 3 and the image of 5.

   (b) $n \longrightarrow (3n) + 1$. The image of 4 is one-half the sum of the images of 3 and 5; that is, their average.

   (c) $n \longrightarrow (3n) - 1$. The images of 3, 4, 5 are consecutive numbers.

   (d) $n \longrightarrow n^2$. The image of 4 is the average of the images of 3 and 5.

   (e) $n \longrightarrow 12 - n$. The images of 3, 4, 5 are in increasing order.

5. For each of the following rules of assignment, choose a set of whole numbers A as the domain of a mapping from A to W having the given rule of assignment. Construct an arrow diagram for each of your mappings. (Note: more than one

answer is possible. Try to choose as "large" a set as you can for the domain A.)

(a) $n \longrightarrow 2n$

(b) $n \longrightarrow \frac{1}{2}n$

(c) $n \longrightarrow n + 2$

(d) $n \longrightarrow n - 2$

(e) $n \longrightarrow (2n) + 3$

(f) $n \longrightarrow (3n) - 2$

6. Study the arrow diagrams below and for each of them answer the following questions as far as they apply.

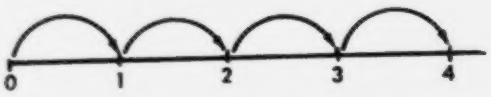
(1) Does the diagram represent a mapping? If not, why not?

(2) If it represents a mapping, what is its domain? Its range?

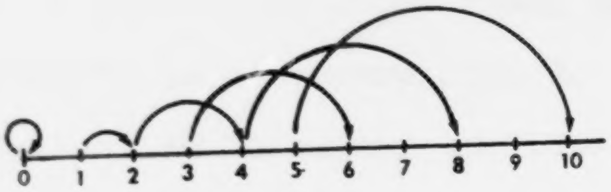
(3) If it is a mapping and it has a rule that is easily expressed in the form $n \longrightarrow ?$ state the rule.

(4) If it is a mapping, is every element of the range the image of exactly one element of the domain?

(a)

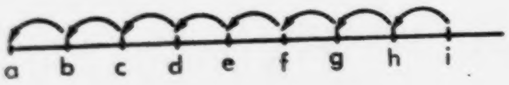


(b)

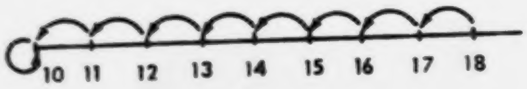


(The arrow at 0 starts and ends at 0.)

(c)

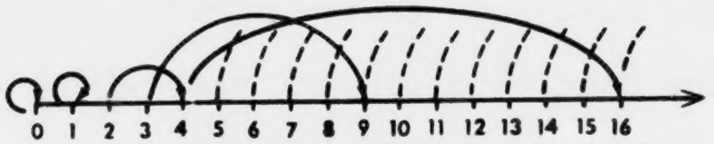


(d)

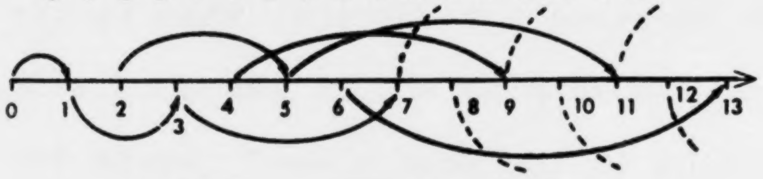


For (e) and (f) let the dotted partial arrow indicate that the domain is the whole numbers, W, and assume a rule that holds for the numbers shown holds for all of W.

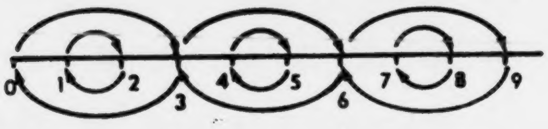
(e)


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

(f)


0 1 2 3 4 5 6 7 8 9 10 11 12 13

(g)


0 1 2 3 4 5 6 7 8 9

7. Make an arrow diagram on a line for each of the following mappings of the given set A to the set of whole numbers, W. Show the image of each number in A, if possible. If this is not possible, show the images of at least five elements of the set A. (You may choose any convenient scale on your number line.)

(a) $n \longrightarrow n + 3$; $A = W$.

(b) $n \longrightarrow (2n) + 1$; $A = W$.

(c) $n \longrightarrow (2n) - 1$; $A = \{1, 2, 3, 4, 5\}$.

(d) $n \longrightarrow n^2 = n \cdot n$; $A = W$.

(e) $n \longrightarrow 3 - n$; $A = \{0, 1, 2, 3\}$.

(f) $n \longrightarrow \frac{1}{2}n$; $A = \{0, 2, 4, 6, 8, 10\}$.

3.5 Mappings of Clock Numbers

In Chapter 1 we studied finite systems consisting of clock numbers and operations on those numbers. In this section we examine mappings for such systems.

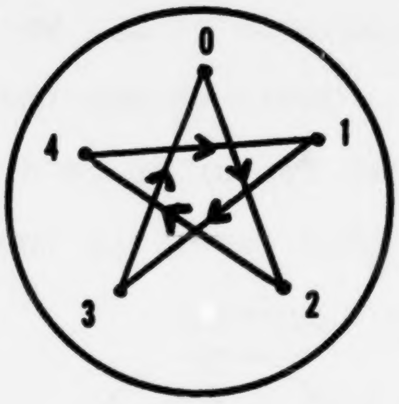


Figure 3.14

Let the domain of a mapping be the set Z_5 of numbers on a clock as shown above and let us map this set to itself by the rule $n \longrightarrow n + 2$ where "+" means addition in $(Z_5, +)$. What is the image of 0? of 2? of 3? What is the domain of this mapping? What is its range? For convenience let us name this mapping h.

Recall the mapping of W to W given by the rule $n \longrightarrow n + 2$. (Of course, "+" in this rule is ordinary addition.) Let us name this mapping f.

Compare the answers to the following questions as each is applied first to f and then to h.

(1) Is the domain of the mapping finite or infinite?
Is the range finite or infinite?

(2) Is the range of the mapping the same as the domain?

(3) Is every element of the range of the mapping the image of exactly one element of the domain?

Now make an arrow diagram on a clock like the one in Figure 3.14 for the mapping of Z_6 to itself given by the rule $n \longrightarrow n - 3$. Let us call this mapping k. You should get the same arrow diagram as for the mapping h. Since the first set and the second set are both Z_6 for the mappings h and k, and since they assign the same images to the elements of Z_6, we see that they are really the same mapping. Thus, we see that the same mapping can be given by rules that appear to be different. You should try to find out why, in this case, the two different rules actually make the same assignments.

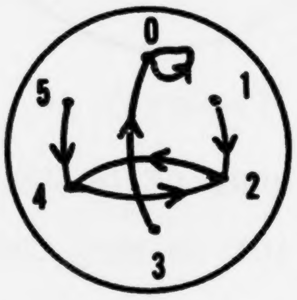


Figure 3.15

Study the mapping (call it s) of the set Z_6 of clock numbers $\{0, 1, 2, 3, 4, 5\}$ by the rule $n \longrightarrow 2n$ (Figure 3.15). Explain why there are two arrows connecting 2 and 4. Notice that there are no arrows with tips at 1, 3, 5. Why do you think this is so?

The mapping t, illustrated in Figure 3.16 maps W to W by the rule $n \longrightarrow 2n$. Explain why there are no arrow tips at 1, 3, 5, 7, and the other odd numbers. Answer the following questions as they apply to s and t.

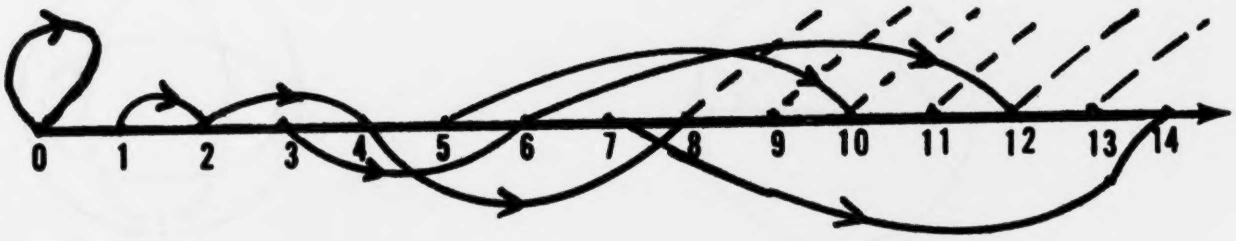


Figure 3.16

(1) What is the domain of the mapping? What is its range?

(2) Is the range the same as the domain?

(3) Is there any whole number that is the image of more than one whole number?

You can answer this last question easily by checking to see whether or not there is any whole number at the tip of more than one arrow.

3.6 Exercises

1. Why do the mappings h and k of Z_5 to Z_5 given by the rules $n \longrightarrow n + 2$ and $n \longrightarrow n - 3$, respectively, turn out to be the same mapping?

2. Study the arrow diagrams below and answer the following questions as they apply to each diagram.

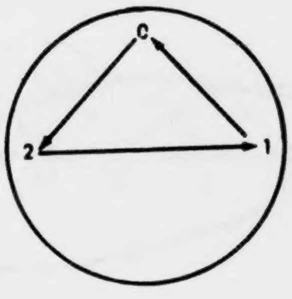
   (1) Does the diagram represent a mapping? If not, why not?

   (2) If it represents a mapping, what are the domain and range?

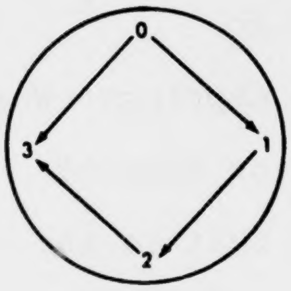
   (3) If it is a mapping and it has an easily expressed rule in the form $n \longrightarrow$? state the rule.

   (4) If it is a mapping, is any clock number in the mapping the image of more than one clock number? If so, which ones?

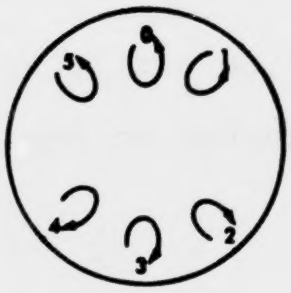
(a)



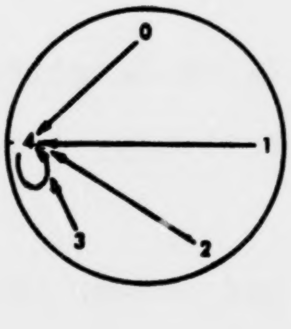
(b)



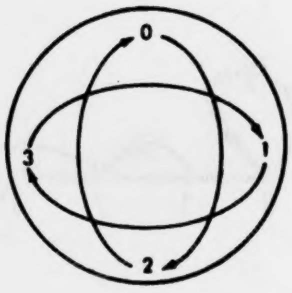
(c)



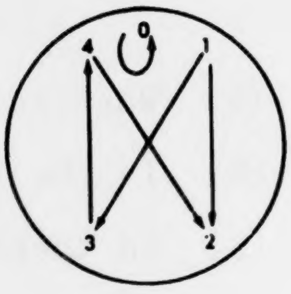
(d)



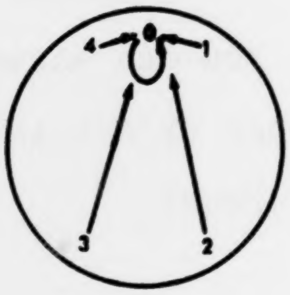
(e)



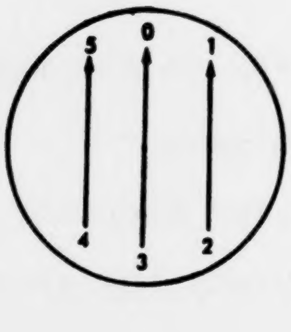
(f)



(g)



(h)



3.7 Sequences

The multiples of 3, that is, 3, 6, 9, 12, ... , considered in the order written are the images in a mapping of N to N given by the rule n ⟶ 3n, where N is the set of natural numbers

{1, 2, 3, 4, 5, ...}. This is but one example of a situation we meet many times in mathematics. That is, we have a set of numbers given in an order, or an ordered set. Another example is 2, 5, 8, 11, ... In this case, as well as in the first, it is possible to think of these numbers as the range of a mapping of N to N. What is the rule for this mapping? Do you see that it is $n \longrightarrow (3n) - 1$? As a third example, consider $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, $\frac{11}{2}$, ... Since these are not natural numbers, they are not images in a mapping of N to N. However, they are images in a mapping of N to a different set of numbers. The rule of this mapping is $n \longrightarrow n + \frac{1}{2}$.

These special mappings, that is, mappings whose domain is N but whose range may be in some other set, are called sequences. The examples given, where the domain is all of N, are called infinite sequences. If the domain is a set of natural numbers from 1 up to some fixed natural number k, the sequence is called finite.

In each of the examples of a sequence given, the range was contained in a set of numbers. This need not be the case. For example, when a teacher records the names of the students in his class in alphabetical order in his register, he constructs a mapping whose domain is the set of natural numbers from 1 up to the natural number which is the number of students in his class. However, even though the range of a sequence may not be a set of numbers, the domain of a sequence must be a set of natural numbers. Since this is the case, we may often omit specific mention of the domain of the sequence and instead merely give the ordered range. We often call this ordered set the sequence.

Below are some other sequences together with the rule of the mapping that determines them.

	Rule	Sequence
(1)	$n \longrightarrow (2n) + 1$	3, 5, 7, 9, 11, 13, 15, 17, ...
(2)	$n \longrightarrow (3n) + 2$	5, 8, 11, 14, ...
(3)	$n \longrightarrow (\frac{1}{2}n) + 2$	$2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$.
(4)	$n \longrightarrow n^2$	1, 4, 9, 16, 25, ...
(5)	$n \longrightarrow (n^2) - n$	0, 2, 6, 12, 20, 30.
(6)	$n \longrightarrow 0$ if n is even $n \longrightarrow 1$ if n is odd	1, 0, 1, 0, 1, 0, 1, 0, 1, 0, ...

In (6), a new kind of rule is used, with two parts. We see that by this rule, for example $17 \longrightarrow 1$ and $26 \longrightarrow 0$.

Another interesting way to construct a sequence is to choose its terms by tossing a coin. For instance, we toss a coin and if the coin comes up "heads," we take 1 for the first term; if the coin comes up "tails" we take 0 for the first term. Then the coin is flipped again and the second term of the sequence is determined in the same way. Thus for the n^{th} term of the sequence, we get 0 or 1 depending on whether the coin comes up "heads" or "tails." This process repeated, say, a hundred times yields a finite sequence.

Some of the examples of sequences given above are finite, and some are infinite. In (5), the domain is $\{1, 2, 3, 4, 5, 6\}$. Thus, this sequence is finite. In (6), the domain is all the natural numbers. Hence this sequence is infinite.

3.8 Exercises

1. (a) What is the domain of each of the sequences in examples (1) - (6)?

   (b) Which of the sequences in examples (1) - (6) are finite and which are infinite?

   (c) How is your answer to (b) related to your answer to (a)?

2. For each of the following sequences the domain is {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Find the range of this sequence using the "coin-flip" rule. Compare your sequence with that obtained by someone else. Would you expect your sequences to be the same? Why?

3. For each of the following sequences the domain is {1, 2, 3, 4, 5, 6} and the range is contained in the set of numbers of arithmetic. Find the range of each sequence with the given rule of assignment.

   (a) $n \longrightarrow n$

   (b) $n \longrightarrow 12 - n$

   (c) $n \longrightarrow (173n) + 312$

   (d) $n \longrightarrow (\frac{5}{17}n) + 37$

   (e) $n \longrightarrow (n^2) + 156$

   (f) $n \longrightarrow [\frac{1}{2}(n^2)] + 79$

4. The rule of a sequence is given as $n \longrightarrow 2n$ if n is odd and $n \longrightarrow \frac{1}{2}n$ if n is even.

   (a) Write down the first 10 numbers in the range of this sequence.

(b) Find the 78th number in the range of this sequence. (By this is meant the image of 78 in the sequence.)

**5. Suppose we start with $1 \longrightarrow 7$ and use the rule that to find the image of 2 we multiply 7, the image of 1, by 3 and then subtract 5 from the product. Thus $2 \longrightarrow (3 \cdot 7) - 5 = 16$. Then we repeat the process with 16, the image of 2, to get the image of 3. That is, $3 \longrightarrow (3 \cdot 16) - 5 = 48 - 5 = 43$. Repeating the process for each natural number in turn, a sequence is obtained in yet a different way.

(a) Write down in order the first 4 numbers in the range of this sequence.

(b) We can describe the way that the images are obtained in this mapping as follows:

(1) $1 \longrightarrow 7 = a_1$

(2) Let a_k represent the image of the natural number k. that is $k \longrightarrow a_k$. Then $k + 1 \longrightarrow [(3a_k) - 5] = a_{k+1}$. Now find a_7 and a_8 given that $a_6 = 1096$.

3.9 Composition of Mappings

Recall the mapping in Example 6 of Section 3.1. Here the domain, set A was {Mary, Steve, Joe, Janet, Peter, Harry}, and the codomain, set B was {Mr. Brown, Mr. Jones, Mr. Ross, Mr. White}. Now, each of these men has a wife, so that we also may have a mapping of A to C, where C is the set of wives {Mrs. Brown, Mrs. Jones, Mrs. Ross, Mrs. White}. The mapping of

A to B we will call f and the mapping of B to C, we will call g. We can draw an arrow diagram for these mappings as in Figure 3.17, with the solid arrows showing the assignments for f and g.

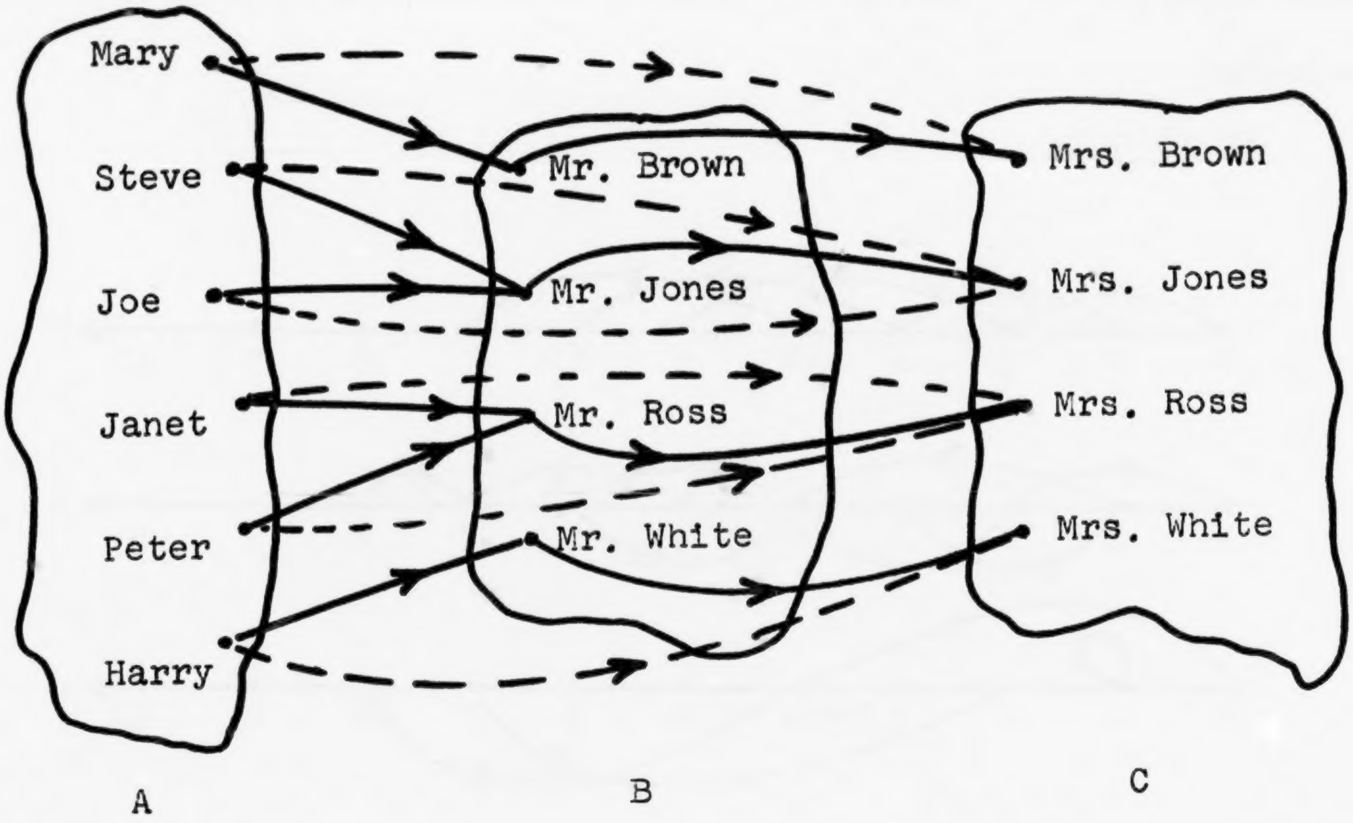
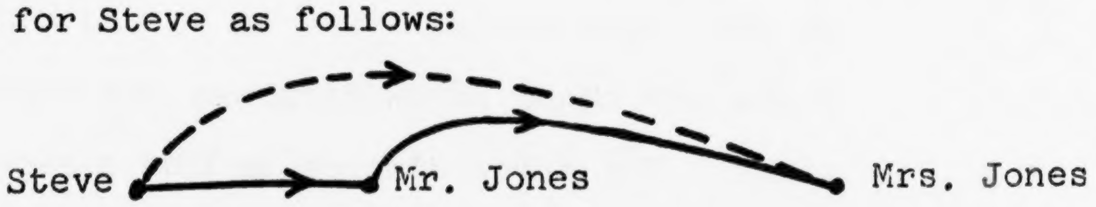

Mary
Steve
Joe
Janet
Peter
Harry
Mr. Brown
Mr. Jones
Mr. Ross
Mr. White
Mrs. Brown
Mrs. Jones
Mrs. Ross
Mrs. White
A
B
C

Figure 3.17

Now to each child in A we can assign a person, the mother. This is done for Steve as follows:


Steve
Mr. Jones
Mrs. Jones

Do you see that this method of assignment assigns to each child in A exactly one person in set C? Because this is true we have constructed a new mapping of A to C from the given mappings f and g. The "dashed" arrows in Figure 3.17 represent this mapping.

To indicate that this new mapping is obtained from the mappings f and g by following the mapping f with the mapping g, we call it "g following f," and write $g \circ f$. This new mapping may also be described as the composition of g with f. Now let us look at some examples of finding the composition mapping from two given mappings.

Example 1.

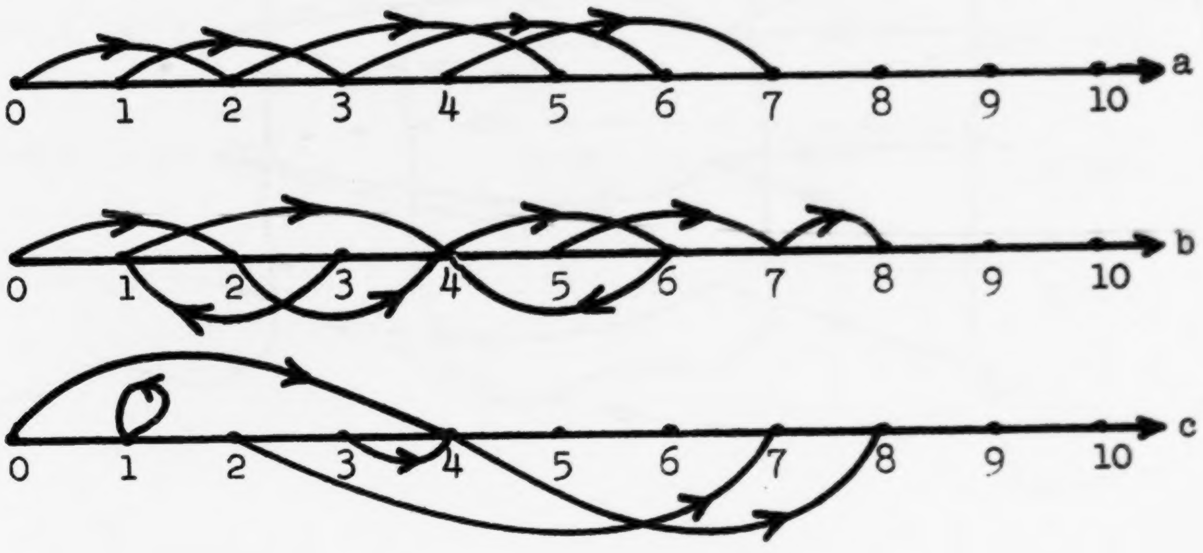


Figure 3.18

On the three number lines a, b, and c in Figure 3.18, are shown arrow diagrams for three mappings. The arrow diagram on line a represents a mapping f of M = {0, 1, 2, 3, 4} to W. The arrow diagram on line b represents a mapping g of Q = {0, 1, 2, 3, 4, 5, 6, 7} to W. The arrow diagram on line c represents the mapping g following f, $g \circ f$, of M to W. It is easy to see how the arrow diagram for g ∘ f is obtained In fact, the process is the same whenever we wish to construct an arrow diagram for the composition of two mappings. We simply take each element in M and apply f, and follow it by g. Thus for the element 0 in M, f maps 0 onto 2; in turn g maps 2 onto 4,

$$\begin{tikzcd}0 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & 2 \arrow[r, "g"] & 4\end{tikzcd}$$

so we draw the arrow from 0 to 4 in the arrow diagram for g ∘ f on line c. Likewise, for the element 3 in M

$$\begin{tikzcd}3 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & 6 \arrow[r, "g"] & 4\end{tikzcd}$$

Continuing in the same way, the arrow diagram for g ∘ f is constructed

In the example above we have used some new notation. Since there are two mappings involved in the problem simply writing $0 \longrightarrow 2$ does not indicate which of the mappings f and g is involved. By writing $0 \xrightarrow{f} 2$ we indicate that *0 is mapped onto 2 by f*, or f maps 0 onto 2, or the image of 0 by f is 2.

Also, the order in which f and g appear in the symbol "$g \circ f$" is chosen for a very specific reason, of which you will become aware later. For the moment, always read $g \circ f$ as "g following f" and remember that the mapping f is applied first.

In this example, note that in order for the composite mapping $g \circ f$ to be meaningful the range of f must be a subset of the domain of g, and this is always the case. Observe that the domain of $g \circ f$ is the domain of f but the range of $g \circ f$ is contained in the range of g.

Example 2.

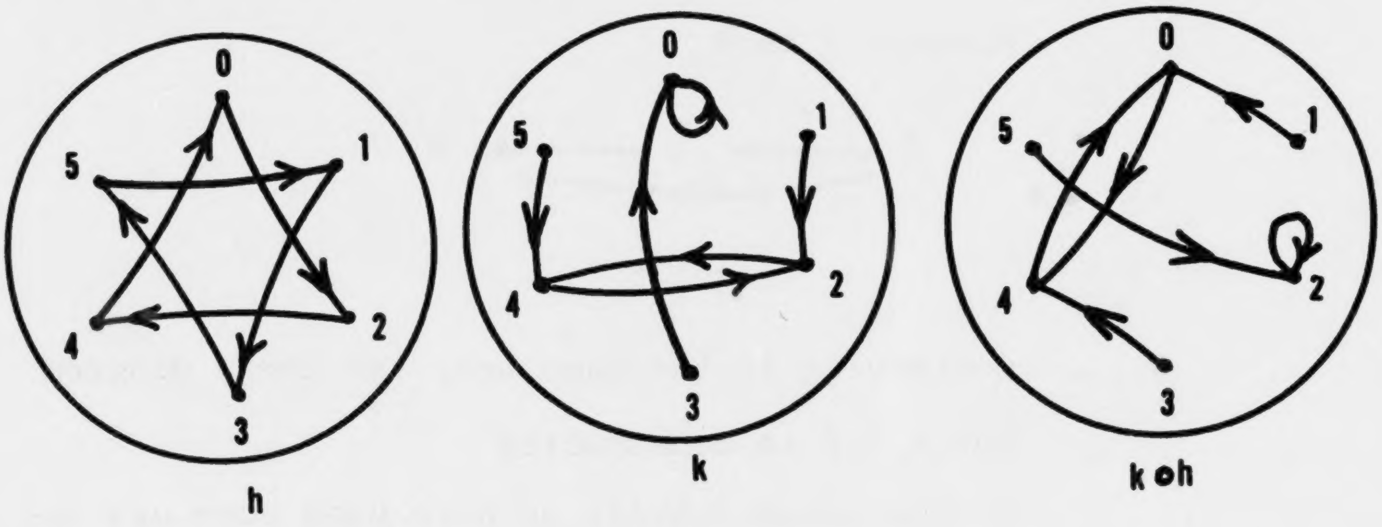


Figure 3.19

In Figure 3.19 h and k are mappings of Z_6 to Z_6 given by the corresponding arrow diagrams. The procedure is the same as in Example 1. Consider 2 in Z_6.

$$\begin{tikzcd}
2 \arrow[r, "h"] \arrow[rr, bend right, "k \circ h"'] & 4 \arrow[r, "k"] & 2
\end{tikzcd}$$

We see that $2 \xrightarrow{k \circ h} 2$. Check the diagram for $k \circ h$ to see whether or not it is correct.

Now take a sheet of paper and construct for yourself the arrow diagram for $h \circ k$. Is $h \circ k$ the same mapping as $k \circ h$? What does this tell you about the importance of the order in composition of mappings?

If you examine carefully the mappings h and k you may be able to find a rule of the form $a \longrightarrow \square$ for each of them. The image of each element of Z_6 by h may be found by "adding 2," and the image of each element in Z_6 by k may be found by "multiplying by 2," where the operations are in $(Z_6, +, \cdot)$. Thus, we may write $n \longrightarrow n + 2$ and $n \longrightarrow 2n$ as rules for h and k, respectively. It is convenient to indicate the mapping associated with the rule by writing "$n \xrightarrow{h} n + 2$" and "$n \xrightarrow{k} 2n$." These are read "the image of n by h is n + 2," and "k maps n to 2n."

In this case we can obtain a rule for $k \circ h$ of the form $n \longrightarrow \square$. To find this rule directly for $k \circ h$, let z be any element of Z_6. Then,

$$\begin{tikzcd}
z \arrow[r, "h"] \arrow[rrr, bend right, "k \circ h"'] & z + 2 \arrow[r, "k"] & 2(z + 2) = & (2z) + 4.
\end{tikzcd}$$

Thus, $n \xrightarrow{k \circ h} (2n) + 4$ is a rule for $k \circ h$.

In the same way, we can find a rule for $h \circ k$.

$$\begin{tikzcd}
z \arrow[r, "k"] \arrow[rr, bend right, "h \circ k"'] & 2z \arrow[r, "h"] & (2z) + 2
\end{tikzcd}$$

Thus, $n \xrightarrow{h \circ k} (2n) + 2$ is a rule for $h \circ k$. Now you can answer the question as to whether $h \circ k = k \circ h$ in a different way by using these rules to find the image of 2 under $h \circ k$ and $k \circ h$. Do you get the same image? If not, then $h \circ k$ and $k \circ h$ are not the same mapping.

Example 3. Two mappings f and g of W to W are given by the rules $n \xrightarrow{f} (2n) + 1$ and $n \xrightarrow{g} 3n$. We shall meet many mappings given in this way and there are several kinds of questions that are commonly asked about such mappings.

(1) Find the image of 27 by f, by g.

$n \xrightarrow{f} (2n) + 1$. Therefore,

$27 \xrightarrow{f} (2 \cdot 27) + 1 = 54 + 1 = 55.$

$n \xrightarrow{g} 3n$. Therefore

$27 \xrightarrow{g} 3(27) = 81.$

(2) List the set of whole numbers, each of which has, by g, the image (a) 51 (b) 103.

(a) $n \xrightarrow{g} 3n$. We need a whole number x such that $3x = 51$. The solution set of this equation is $\{17\}$.

(b) The solution set of the equation $3x = 103$ in $(W,+,\cdot)$ is $\emptyset$ or $\{\}$.

(3) What is the image of 5 by (a) $g \circ f$ and (b) $f \circ g$?

(a) $n \xrightarrow{f} (2n) + 1$ and $n \xrightarrow{g} 3n$. Therefore $5 \xrightarrow{f} (2 \cdot 5) + 1 = 10 + 1 = 11$, and $11 \xrightarrow{g} 3 \cdot 11 = 33$. We have, then $5 \xrightarrow{f} 11 \xrightarrow{g} 33$ so that $5 \xrightarrow{g \circ f} 33$.

(b) $5 \xrightarrow{g} 3 \cdot 5 = 15$, and $15 \xrightarrow{f} (2 \cdot 15) + 1 = 30 + 1 = 31$. Hence, $15 \xrightarrow{f \circ g} 31$.

If we wish to find many images by $f \circ g$ or by $g \circ f$ it is more efficient to first find rules of the form $n \longrightarrow \square$ for $f \circ g$ and $g \circ f$. This is done in the same way as in Example 2. We begin by letting w represent a whole number. Then

$$w \xrightarrow{f} (2w) + 1 \xrightarrow{g} 3(2w + 1) = (3(2w)) + (3 \cdot 1)$$
$$= (6w) + 3$$

Thus, a rule for $g \circ f$ is $n \xrightarrow{g \circ f} (6n) + 3$.

Also,

$$w \xrightarrow{g} 3w \xrightarrow{f} (2(3w)) + 1 = (6w) + 1$$

Thus, a rule for $f \circ g$ is $n \xrightarrow{f \circ g} (6n) + 1$.

Using these rules,

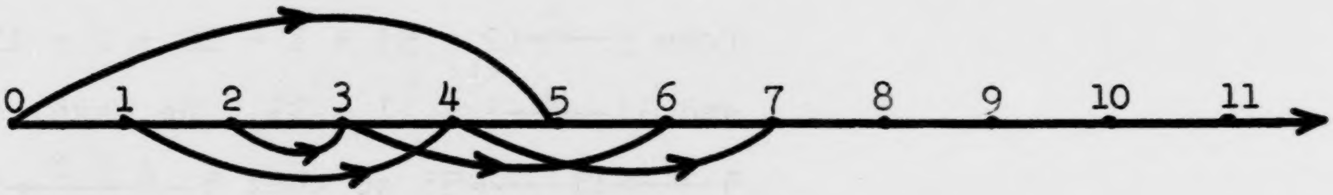
$$5 \xrightarrow{g \circ f} (6 \cdot 5) + 3 = 30 + 3 = 33, \text{ and}$$

$$5 \xrightarrow{f \circ g} (6 \cdot 5) + 1 = 30 + 1 = 31.$$

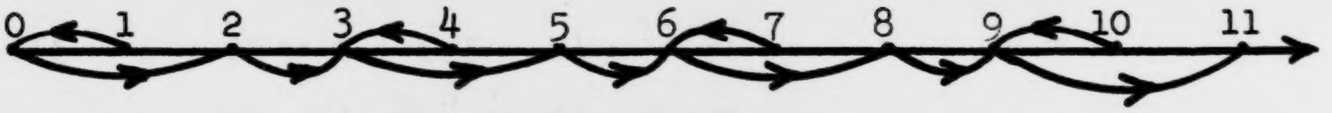
3.10 Exercises

1. Two mappings f and g are given by the following arrow diagrams. In each mapping the codomain is W.

f:



g:



(a) Find the image of 3 by f.

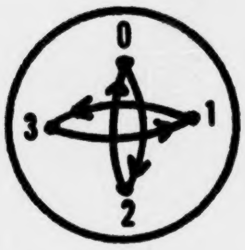
(b) Find the image of 3 by g.

(c) Find the image of 3 by $g \circ f$.

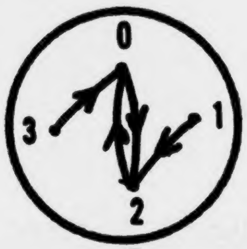
(d) Draw an arrow diagram on a line for $g \circ f$.

(e) If possible, draw an arrow diagram on a line for $f \circ g$. If it is not possible, what goes wrong?

2. Two mappings f and g of Z_4 to Z_4 are given by the following arrow diagrams:



f



g

(a) Construct an arrow diagram for $g \circ f$.

(b) Construct an arrow diagram for $f \circ g$.

(c) Is $f \circ g$ the same mapping as $g \circ f$? Why?

In questions 3 and 4 f, g, and h are mappings of W to W given by the following rules:

$n \xrightarrow{f} 3n$, $\quad n \xrightarrow{g} n + 2$, $\quad n \xrightarrow{h} (2n) + 1$.

3. Find the image of 67

(a) by f	(d) by $f \circ g$	(g) by $g \circ f$
(b) by g	(e) by $f \circ h$	(h) by $h \circ f$
(c) by h	(f) by $g \circ h$	(i) by $h \circ g$.

4. Find a whole number, or explain why there is none

(a) whose image by g is 13.

(b) whose image by h is 101.

(c) whose image under $f \circ h$ is 33.

(d) whose image under $g \circ f$ is 14.

(e) whose image under $f \circ h$ is 12.

5. Let r be the mapping of Z_{12} to Z_{12} given by the rule $n \xrightarrow{r} 4n$.

(a) Find the image of 7 by r.

(b) Find the image of 10 by r.

(c) List the set of elements of Z_{12} each of which has by r, the image

(1) 4
(2) 6
(3) 8
(4) 3.

3.11 Inverse and Identity Mappings

You learned in arithmetic if you add 5 to a certain number and then subtract 5 from the sum, you end at the same number with which you began. In other words the effect of adding 5 is nullified by subtracting 5. Similarly the effect of multiplying by a number is nullified by division by that same number. This suggests the question: Is there for each mapping another, such that when one is followed by the other the effect of the first is nullified by the second? It is easy to see that this may be the case by looking at an example.

Example 1.

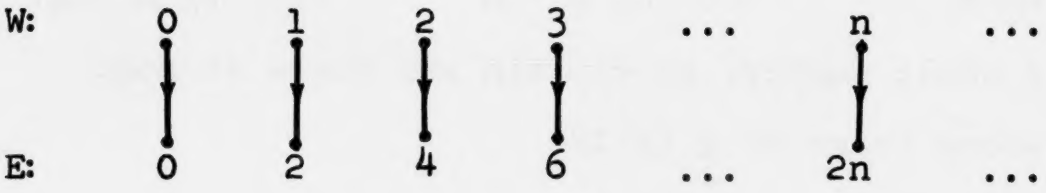


Figure 3.20

Here the first set, or domain, is W and the codomain is the set E = {2, 4, 6, 8, ...}. It is easy to see, calling this mapping f, that the range of f is all of the set E, since each even number would be at the tip of an arrow. Now, to nullify the effect of f we must carry each image back to its source. In terms of an arrow diagram this means that each arrow in the diagram must be reversed, as shown in Figure 3.21.

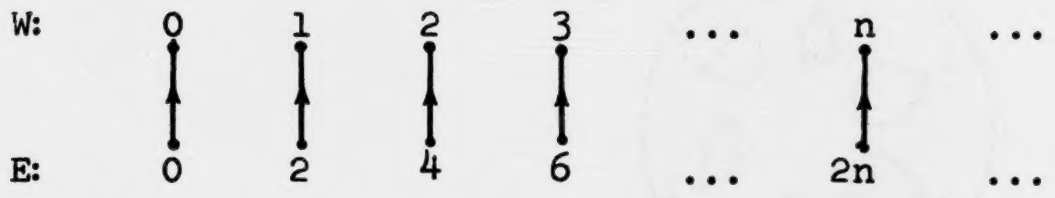


Figure 3.21

Is this an arrow diagram for a mapping of E to W? Since each number in E is assigned exactly one whole number the answer is "yes." Call this mapping g. Its rule is $n \xrightarrow{g} \frac{1}{2}n$. Thus we have two mappings f of W to E, and g of E to W such that g nullifies the effect of f. When we say that g nullifies the effect of f, we mean that $g \circ f$ maps W to W and that each element n of W is mapped onto itself. That is, $n \longrightarrow n$ is the rule for $g \circ f$. It is easy to see that this is the case for, if $n \xrightarrow{f} 2n$, then $2n \xrightarrow{f} \frac{1}{2}(2n) = n$ so that $n \xrightarrow{g \circ f} n$. Whenever we have a set A given it is possible to construct the mapping j of A to A with the rule $n \longrightarrow n$. For example, the arrow diagram for the mapping j of Z_6 to Z_6 is shown in Figure 3.22. Each element of Z_6 is mapped onto itself.

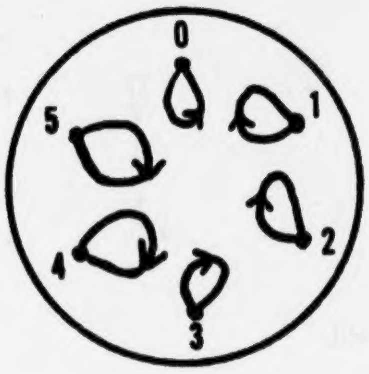


Figure 3.22

An incomplete arrow diagram on a line for this mapping j of W to W is shown in Figure 3.23.

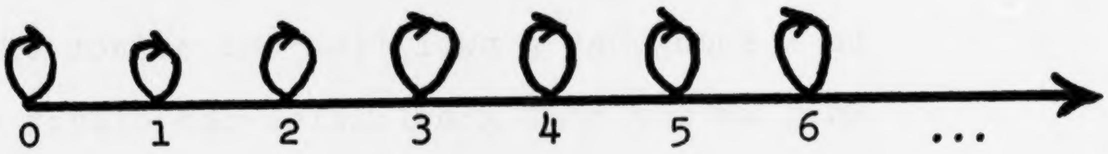


Figure 3.23

It is clear that the mapping j of Z_6 to Z_6 is not the same as the mapping j of W to W even though they have the same rule, $n \longrightarrow n$. To indicate that we are talking about the identity mapping on a given set A, we may write "j_A" for that particular identity mapping. Thus, the identity mapping on W is written "j_W."

Now we can describe the situation in our first example more economically. To say that g "nullifies" f is to say that $g \circ f$ is the identity mapping on W. That is $g \circ f = j_W$.

Is f also the inverse mapping to g? That is, does $f \circ g = j_E$? Now, for any element n in E, $n \xrightarrow{g} \frac{1}{2}n$, which is a whole number, since n must be even to be a number in E. Then $\frac{1}{2}n \xrightarrow{f} 2(\frac{1}{2}n) = n$. Thus we have $n \xrightarrow{f \circ g} n$ as a rule for the mapping $f \circ g$ of E to E. This means that $f \circ g = j_E$ and f is the inverse mapping to g.

Does every mapping have an inverse? Consider the following example.

Example 2. h is a mapping of Z_6 to Z_6 given by the rule $n \xrightarrow{h} 2n$. An arrow diagram for this mapping is shown in Figure 3.24.

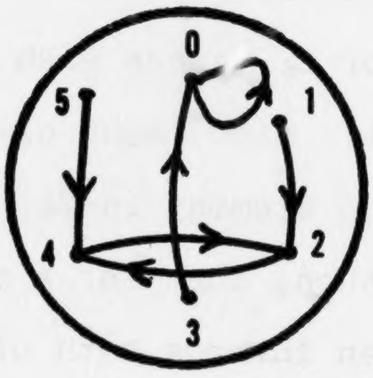


Figure 3.24

(This diagram should be an old friend by now.) We could begin by reversing the direction of each arrow in the diagram to "nullify" the effect of h. The following arrow diagram results:

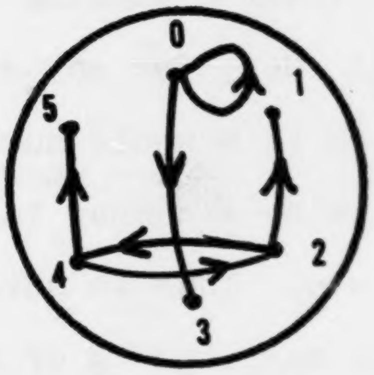


Figure 3.25

But we see that this assignment is not a mapping, for 0 is assigned 0 and 3, 2 is assigned 1 and 4, 4 is assigned 2 and 5. Furthermore, 1, 3, and 5 are each assigned no element in Z_6. What went wrong? If we look carefully at Figure 3.24 we see that each of 0, 2 and 4 is the image of two elements of Z_6. since each has two arrows pointing to it. Also, each of 1, 3 and 5 is the image of no element in Z_6.

We conclude, then, that for a mapping f of A to B to have an inverse both of the following must hold:

(1) Each element of B must be the image of an element of A. In an arrow diagram this means that every element in B is at the tip of an arrow. Thus, the range of f is all of B. We say then that f maps A onto B.

(2) Each image in the mapping must be the image of only one element of A. This means that in an arrow diagram no element of B is at the tip of more than one arrow. We say then that f is a one-to-one mapping of A to B.

Thus, in order for a mapping of A to B to have an inverse, it must be a one-to-one mapping of A onto B.

3.12 Exercises

1. Determine whether or not each of the following mappings has an inverse mapping. In each case which of the conditions (1) and (2) above holds or does not hold, and why.
   (a) The mappings in Examples 2, 4, 5 and 6 of Section 3.1.
   (b) The mapping f of Z_5 to Z_5 given by the rule $n \xrightarrow{f} 3n$.
   (c) The mapping g of Z_6 to Z_6 given by the rule $n \xrightarrow{g} n + 3$.
   (d) The mapping h of Z_6 to Z_6 given by the rule $n \xrightarrow{h} 3n$.
   (e) Let A be the set of living persons on earth and let B be the set of countries. Consider the mapping defined by assigning to each person the country in which he lives.
2. For each mapping in Exercise 1 that has an inverse, de-

scribe that inverse and if possible, give a rule of the form $n \longrightarrow ?$.

3. Let f be the mapping of W to W with the rule $n \longrightarrow 2n$. Let j be the identity mapping of W to W.
   (a) Show that the compositions j following f and f following j are the same as f.
   (b) Suppose you do not know the rule for f. Do you think that the compositions $j \circ f$ and $f \circ j$ are the same as F? Why?
4. Make an arrow diagram of the identity mapping of the set of numbers $\{0, 1, 2, 3\}$ onto itself.
5. (a) Make an arrow diagram on a line of the mapping h of W onto $R = \{2, 3, 4, \ldots\}$ with the rule $n \longrightarrow n + 2$.
   (b) Make an arrow diagram for the inverse mapping k.
   (c) Show by an arrow diagram that $k \circ h = j_W$.
   (d) Show by an arrow diagram that $h \circ k = j_R$.
6. The rule of a mapping of W to W is $n \longrightarrow (3n) + 2$. To find the image of a given number you perform two operations.
   (1) multiply the given number by 3. $(3n)$
   (2) add 2 to the product. $((3n) + 2)$

   What is R, the range of the mapping?

   To find a number given its image, you reverse these operations and the order.

   (1) subtract 2 from the given image. $(n - 2)$
   (2) divide the difference by 3. $\frac{(n - 2)}{3}$

   Since every mapping is onto its range R, and each number in the range R of this mapping is the image of only one number in the domain W, $n \longrightarrow \frac{n - 2}{3}$ is a rule for the inverse mapping R to W of the mapping W to R with rule $n \longrightarrow (3n) + 2$. (The domain R of the inverse mapping is {2, 5, 8, 11, ...}.) For each of the following rules find a domain D and a range R consisting of whole numbers, so that the rule constitutes a one-to-one mapping of D onto R. Then find the rule of the inverse mapping of R onto D.

(a) $n \longrightarrow (2n) + 1$ (b) $n \longrightarrow (3n) - 2$

(c) $n \longrightarrow n - 2$ (d) $n \longrightarrow n + 2$

(e) $n \longrightarrow (48n) + 25$ (e) $n \longrightarrow (8n) + 1800$

7. Make an arrow diagram of the mapping of the set of clock numbers Z_8 to itself for each of the following rules, and determine whether it has an inverse. Explain in each case why the mapping does or does not have an inverse. If the mapping has an inverse give a rule for the inverse mapping.

(a) $n \longrightarrow n + 1$ (b) $n \longrightarrow n - 3$

(c) $n \longrightarrow 3$ (d) $n \longrightarrow (2n) + 1$

3.13 Special Mappings of W to W

Among the various kinds of mappings of W to W that we have looked at so far there are two that deserve special attention. One of these is the class of mappings which have rules such as $n \longrightarrow n + 4$, $n \longrightarrow n + 13$, $n \longrightarrow n + 137$, etc. We may describe this class of mappings as those mappings of W to W which have a rule of the type $n \longrightarrow n + a$ for a fixed whole number a.

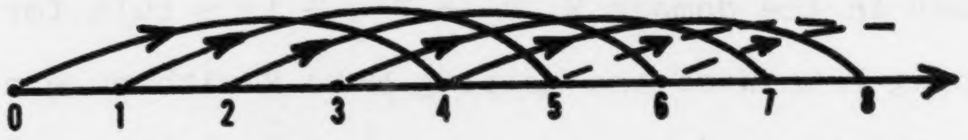


Figure 3.26

In the above (incomplete) arrow diagram on a line for the mapping of W to W with the rule n⟶n + 4, we see that each numbered point on the line is assigned a point 4 units to the right. We could represent this mapping by two "slide rules" as shown in Figure 3.27, where the upper slide rule has been moved over 4 units to the right.

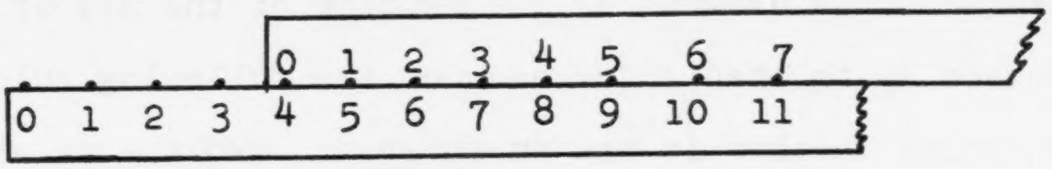


Figure 3.27

This interpretation of the mapping of W to W with the rule n⟶n + 4 suggests that each assignment of an image may be thought of as a "jump" or a "move" 4 units to the right. Another way to look at this mapping is to draw two parallel number lines using the same scale, as shown in Figure 3.28, then drawing the arrows from one line to the other.

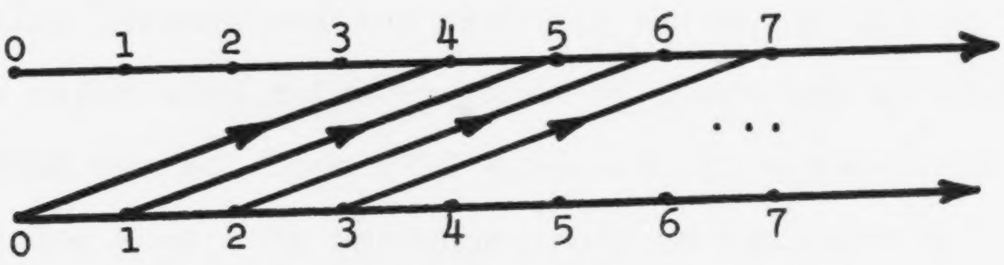


Figure 3.28

Now let us look at the composition of two such mappings and consider whether these compositions have any special properties as mappings of W to W. Let f and g be mappings of W to W with the rules $n \xrightarrow{f} n + 1$ and $n \xrightarrow{g} n + 3$.

The rule for $g \circ f$ is found as follows, where w represents any whole number:

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[rrr, bend right, "g\circ f"'] & w + 1 \arrow[r, "g"] & (w + 1) + 3 = w + (1 + 3) = & w + 4 \end{tikzcd}$$

Thus $n \xrightarrow{g \circ f} n + 4$ is a rule for $g \circ f$ and the composition of the mappings f and g is again a mapping of the same kind. In an arrow diagram for this mapping, each point of the number line is mapped onto a point 4 units to the right.

An interesting diagram for $g \circ f$ results if we construct the diagram on three parallel number lines a, b, c as in Figure 3.29.

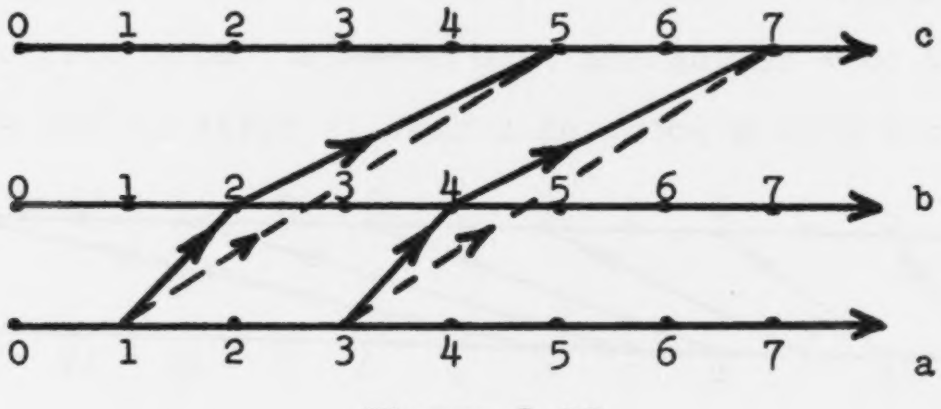


Figure 3.29

In the diagram we have located the image of 1 and the image of 3 by $g \circ f$. Since $1 \xrightarrow{f} 1 + 1 = 2$ we draw an arrow from 1 on line a to 2 on line b. Then $2 \xrightarrow{g} 5$ so we draw an arrow from 2 on line b to 5 on line c. To indicate that $1 \xrightarrow{g \circ f} 5$ we draw an arrow from 1 to line a to 5 on line c. The same process is shown for the image of 3 by g o f. If we were to fill in all the arrows, f would be represented by an arrow diagram from line a to line b, g by an arrow diagram from line b to line c, and g o f by an arrow diagram from line a to line c. Again, we see that g o f maps each whole number onto a point 4 units to the right on line c.

You should check to see whether or not f o g is a mapping of the same kind as f and g. Also is f o g = g o f?

A second special class of mappings are those which have rules such as $n \longrightarrow 3n$, $n \longrightarrow 47n$, $n \longrightarrow 1309n$, etc. We may describe this class of mappings as those mappings of W to W which have a rule of the form $n \longrightarrow an$, for a fixed non-zero whole number a.

In Figure 3.30 we show an arrow diagram from one number line a to a parallel number line b with the same scale for the mapping of W to W having the rule $n \longrightarrow 2n$. Note that each point on a is mapped onto a point on b that is twice as far away from 0.

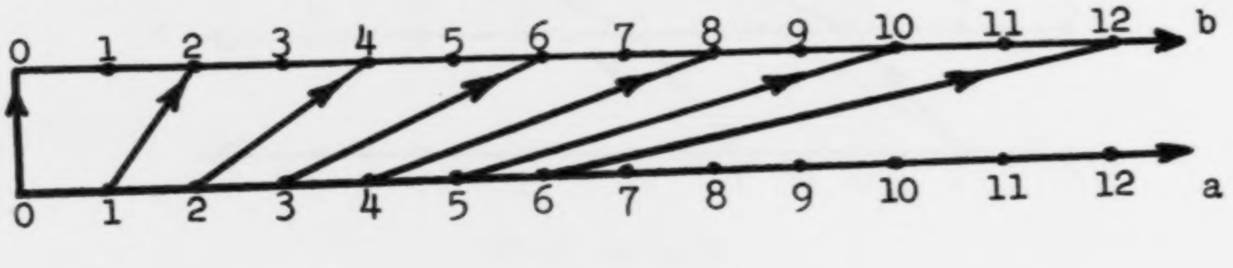


Figure 3.30

Let f and g be two mappings of this type given by rules $n \xrightarrow{f} 3n$ and $n \xrightarrow{g} 5n$. We leave it to you as an exercise to answer the following questions:

(1) Are f o g, g o f mappings of the same kind as f and g?

(2) Does f o g = g o f?

Each of the special mappings we have considered has a particular interpretation that is important in many applications of mappings. We have seen that a mapping of W to W having a rule such as $n \longrightarrow n + 7$ may be interpreted as a "move" 7 units to the right on the number line. Since we can move 7 units to the right on the line, we would expect to be able to nullify this move by a move 7 units to the left on the line. But this would require a rule $n \longrightarrow n - 7$, which cannot be a rule for a mapping of W to W. Why not? Thus, this mapping has no inverse.

Looking at the question another way, suppose we pick the point 5 on the line. We see that the range of the mapping given above is not all of W, since there is no whole number x such that $x \longrightarrow x + 7 = 5$.

In general, a mapping of W to W with a rule $n \longrightarrow n + a$ has no inverse if a is greater than 0. A parallel problem is solving the equation $x + a = b$, which arises in trying to find a number whose image is b in a mapping with the rule $n \longrightarrow n + a$. These problems, and their solutions, will be considered in Chapter 4.

We observed that a mapping of W to W with a rule such as $n \longrightarrow 7n$ may be interpreted on a number line as mapping each point to a point 7 times as far from the 0 point.

As above, we can see that a mapping of W to W having a rule such as $n \longrightarrow 7n$ does not have an inverse since the range is not all of W. For example, there is no whole number x whose image is 24. The parallel problem in terms of equations is that of solving the equation $7x = 24$. These problems, and their solutions, will be considered in Chapter 12.

3.14 Exercises

1. For each of the following mappings of W to W:

   (a) find the range of the mapping;

   (b) state whether or not the mapping is one having a rule of the type $n \longrightarrow n + a$ or $n \longrightarrow an$ and if so, give the value of a;

   (c) draw an arrow diagram for the mapping on two parallel lines. (Choose your scale carefully.)

   (1) $n \longrightarrow n + 7$
   (2) $n \longrightarrow (2n) + 1$
   (3) $n \longrightarrow 5n$
   (4) $n \longrightarrow (4n) + 3$
   (5) $n \longrightarrow n + 25$
   (6) $n \longrightarrow 60n$

2. Consider the mapping h of the clock numbers Z_{12} to Z_{12} given by the rule $n \longrightarrow n + 7$.

   (a) Is every number of Z_{12} an image in this mapping; that is, is h a mapping of Z_{12} onto Z_{12} ?

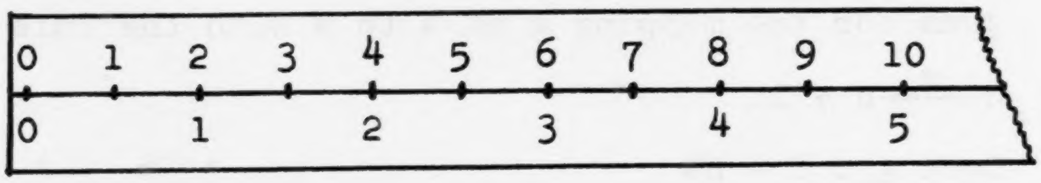
   (b) Is any number of Z_{12} an image for more than one clock number; that is, is h a one-to-one mapping of Z_{12} to Z_{12} ?

   (c) Does the mapping h have an inverse mapping g so that $g \circ h = j$, the identity mapping on Z_{12}? How is your answer related to your answer to parts (a) and (b) of this question?

   (d) Can you describe this mapping in terms of "moves" on the face of the clock? Illustrate your answer on a drawing of a clock face.

   (e) For each "move" in this mapping, is there a "move" that nullifies it? Is there a mapping of Z_{12} to Z_{12} that corresponds to the nullifying "moves"? If so, what is a rule for this mapping?

3. Below is a slide rule arrangement for the mapping of W into W with the rule $n \longrightarrow 2n$. Notice that the lower ruler is scaled by a unit that is twice as long as the unit of the upper ruler.



(a) How would you place the lower ruler so as to make a slide rule arrangement for $n \longrightarrow (2n) + 1$?

(b) Make a slide rule arrangement for each of the mappings of W to W given by the rules

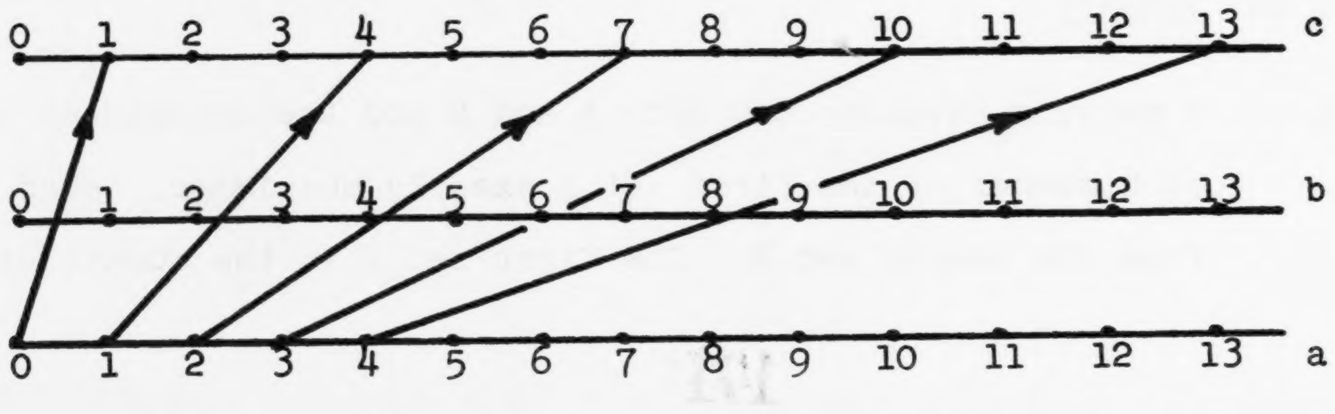
(1) $n \longrightarrow n + 5$ (3) $n \longrightarrow (4n) + 5$

(2) $n \longrightarrow 4n$ (4) $n \longrightarrow (4n) + 20$

4. Using three parallel number lines a, b, and c, as in Figure 3.29, find the arrow diagrams for g o f if f and g are mappings of A = {0, 1, 2, 3, 4, 5} to W given by the rules

(a) $n \xrightarrow{f} n + 6$ and $n \xrightarrow{g} 7n$

(b) $n \xrightarrow{f} 4n$ and $n \xrightarrow{g} n + 2$


0 1 2 3 4 5 6 7 8 9 10 11 12 13 c
0 1 2 3 4 5 6 7 8 9 10 11 12 13 b
0 1 2 3 4 5 6 7 8 9 10 11 12 13 a

5. The figure above shows an incomplete arrow diagram, on lines a and c, of the mapping h of W to W given by the rule $n \xrightarrow{h} (3n) + 1$. Copy this figure on your paper.

   (a) Construct, on lines a and b, an incomplete arrow diagram for the mapping f of W to W given by the rule $n \xrightarrow{f} 3n$.

   (b) Construct, on lines b and c, an incomplete arrow diagram for the mapping g of W to W with the rule $n \xrightarrow{g} n + 1$.

   (c) Does $g \circ f = h$?

   (d) Given any mapping f of W to W with a rule like that of h, that is, of the form $n \xrightarrow{f} (pn) + q$ for p and q fixed whole numbers, $p \neq 0$. Can you write f as the composition of two mappings of W to W having rules of the form $n \longrightarrow an$ and $n \longrightarrow n + b$ for fixed whole numbers a and b, $a \neq 0$? If so, how would you do so for the mapping f of W to W given by the rule $n \longrightarrow (77n) + 1306$?

   (e) Try to give an argument to show that the composition of any two mappings of W to W having rules of the form $n \longrightarrow an + b$ and $n \longrightarrow n + b$ will have a rule of the form $n \longrightarrow (pn) + q$.

3.15 Summary

1. A mapping involves two sets A and B and the assignment to each member of the first set A exactly one image, taken from the second set B. The first set A is the domain of

the mapping, and the set of images is the range of the mapping. The domain is all of the first set A but the range may be only part of the second set B.

2. Many mappings are given by a rule involving arithmetic operations. In this chapter we considered mappings of A to B, for A and B sets of whole numbers. Many mappings were given by rules of the form $n \longrightarrow (an) + b$, where a and b are fixed numbers in each mapping and n takes on whole number values. For a particular mapping f of A to B the rule is often written as $n \xrightarrow{f} (an) + b$.

3. For a mapping of A to W given by a rule we may also construct other representations of the mapping. Among these are arrow diagrams, arrow diagrams on a number line, arrow diagrams on a "clock," arrow diagrams between parallel number lines, charts, and tables. On the other hand, a mapping may be given by one of these representations for which there is no easily seen rule or method of assignment other than that given by the diagram, table or chart.

4. Whenever the range of a mapping f is contained in the domain of a mapping g we can construct the mapping g o f, (g following f or the composition of g with f) by first applying f to each element in the domain of f and then applying g to the image. The domain of g o f is the domain of f. The range of g o f is contained in the range of g.

5. In order that a mapping f of A to B have an inverse f must map A onto B and f must be a one-to-one mapping of A to B. Since every mapping may be considered as a mapping

onto its range R, a one-to-one mapping f of A to R has an inverse mapping g of R to A.

6. Mappings of W to W having rules of the type $n \longrightarrow n + b$ may be interpreted as "moves" to the right on the number line. Since they are not onto W (except for $b = 0$), they do not have inverse mappings. However, the composition of any two such mappings is a mapping of the same kind.

7. Mappings of W to W having rules of the type $n \longrightarrow an$, $(a \neq 0)$ may be interpreted as "stretching" the number line. These also are not onto W (except for $a = 1$) and do not therefore have inverse mappings. However, the composition of any two such mappings is a mapping of the same kind.

8. Any mapping of W to W with a rule $n \longrightarrow (an) + b$ can be considered as the composition of a mapping with rule $n \longrightarrow an$ followed by the mapping with rule $n \longrightarrow n + b$.

3.16 Review Exercises

1. Let f and g be mappings of W to W with rules $n \xrightarrow{f} n + 3$ and $n \xrightarrow{g} 2n$.
   (a) Make an arrow diagram on a number line for f, g and g o f.
   (b) Find rules for g o f and f o g of the form $n \longrightarrow ?$.
   (c) Find the image of 637 by g o f and f o g. Are the two images the same? Does f o g = g o f? Why?
   (d) Find the set of values $\underline{x}$ such that the image of $\underline{x}$ by g o f is 24.

(e) Find the set of values x such that the image of x by $f \circ g$ is 24.

2. Let a and b be parallel number lines scaled with the same unit and in the same direction.

   (a) Make an arrow diagram from a to b for the mapping f of W to W with the rule $n \longrightarrow n + 4$.

   (b) Is f an onto mapping? Why?

   (c) Is f a one-to-one mapping? Why?

3. Draw an arrow diagram from number line a to number line a' for each of the following mappings of W to W with the lines a and a' drawn as indicated.

   (a) a is parallel to a'; the lines are scaled with the same unit with the same direction; the rule is $n \longrightarrow 2n - 1$.

   (b) a is parallel to a'; the unit scale on a is twice as long as the unit scale on a'; the lines are scaled in the same direction with zero points opposite each other; the rule is $n \longrightarrow 2n$.

   (c) a intersects a' at point A; the lines are scaled with the same unit from A; the rule is $n \longrightarrow 2n$.

   (d) a is parallel to a'; the lines are scaled with the same unit in opposite directions; the rule is $n \longrightarrow n + 2$.

4. Make an arrow diagram for each of the following mappings where Z_n is the set of n clock numbers and the operations are the clock operations, + and ·.

(a) From Z_5 to Z_5 with the rule $n \longrightarrow 2n + 1$.

(b) From Z_4 to Z_4 with the rule $n \longrightarrow 2n$.

(c) Which of the mappings in (a) and (b) are one-to-one? Is either an "onto" mapping? Do you now know that one of these mappings has an inverse? Why? Find the rule of the inverse mapping.

CHAPTER 4

THE INTEGERS AND ADDITION

4.1 Introduction

In Chapter one, we studied the system $(Z_7,+)$, and also worked with equations in that system. For example, the solution set of

$$6 + x = 1$$

in $(Z_7,+)$ is $\{2\}$ since

$$6 + 2 = 1$$

is a true sentence in $(Z_7,+)$. Study the following examples of equations and their solution sets in $(Z_7,+)$.

Equation	Solution Set in $(Z_7,+)$
$3 + x = 5$	$\{2\}$
$5 + x = 3$	$\{5\}$
$x + 6 = 2$	$\{3\}$
$x + 2 = 6$	$\{4\}$

Each of these equations has a solution in $(Z_7,+)$; the solution set is not empty. In fact, if you choose any two elements a and b from the set Z_7, then the equation $x + a = b$ has a solution in $(Z_7,+)$. In other words, in $(Z_7,+)$ it is always possible to solve an equation of the type "$x + a = b$."

Also in Chapter one, we worked with equations in the system $(W,+)$. Some examples of such equations are listed below.

Equation	Solution Set in $(W,+)$
$5 + x = 14$	$\{9\}$
$x + 21 = 42$	$\{21\}$
$x + 98 = 103$	$\{5\}$

However, look at the equation "$6 + x = 1$." In $(W,+)$, this equation has no solution; the solution set is empty. There is no whole number which may be added to 6 to produce 1.

From this example, we see an important difference between the systems $(Z_7,+)$ and $(W,+)$. If a and b are elements of Z_7, we know that the equation "$x + a = b$" has a solution in $(Z_7,+)$. But if a and b are elements of W, the equation "$x + a = b$" may not have a solution. (As we have just seen, if a is 6 and b is 1, there is no whole number solution.) Study the following examples which help to make this difference between the two systems clear.

Equation	Solution Set in $(Z_7,+)$	Solution Set in $(W,+)$
$5 + x = 6$	$\{1\}$	$\{1\}$
$6 + x = 5$	$\{6\}$	$\{\ \}$
$x + 2 = 5$	$\{3\}$	$\{3\}$
$x + 5 = 2$	$\{4\}$	$\{\ \}$

4.2 Exercises

1. Find the solution set of each of the following equations in $(Z_7,+)$:

(a) $3 + x = 0$ (d) $6 + x = 6$ (g) $x + 1 = 0$

(b) $x + 5 = 1$ (e) $a + 5 = 2$ (h) $3 + x = 1$

(c) $n + 3 = 6$ (f) $a + 2 = 5$ (i) $y + 4 = 1$

2. Find the solution set of each of the following equations in $(Z_3,+)$:

(a) $x + 1 = 0$ (c) $x + 2 = 0$

(b) $2 + x = 1$ (d) $x + 1 = 2$

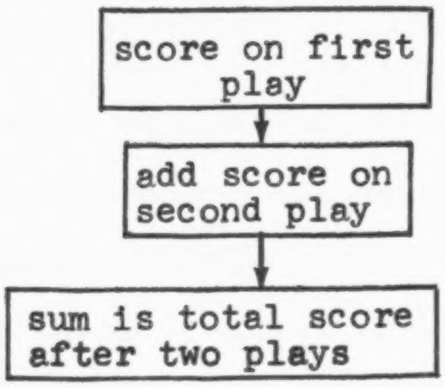
3. Does every equation of the type "$x + a = b$" have a solution in $(Z_3,+)$?
4. Make up five different equations of the type "$x + a = b$" in the system $(Z_4,+)$, and find the solution set of each.
5. Does every equation of the type "$x + a = b$" have a solution in $(Z_4,+)$?
6. Find the solution set of each of the following equations in $(W,+)$:

(a) $x + 8 = 19$
(b) $25 + x = 44$
(c) $44 + x = 25$
(d) $x + 101 = 213$
(e) $x + 213 = 101$
(f) $x + 17 = 39$
(g) $x + 53 = 1006$
(h) $97 + x = 408$
(i) $408 + x = 97$

7. Make up five equations of the type "$x + a = b$" which have solutions in $(W,+)$. Then make up five equations of the type "$x + a = b$" which do not have solutions in $(W,+)$. In both cases, use equations which are different from those in Exercise 6.

4.3 Some New Numbers

There are many games in which you may either win or lose points. Suppose that you are playing such a game, and on the first play you win 6 points. You agree that you will add your score on the second play to the 6 points you already have in order to obtain your total score at the end of two plays. That is, if x represents your score on the second play of the game, then



$$6 + x$$

is your total score at the end of two plays. Now suppose that at the end of the second play your total score is 1; that is,

$$6 + x = 1.$$

Did you win or lose on the second play? It is rather easy to see that you must have lost 5 points. But having agreed to add the second score to the first to get the total score, then x must be a number which added to 6 produces 1. Since there is no whole number that will do this, we shall create a new number called

negative five

and written $^{-}5$.

This is to be a number which, added to 6 gives the sum 1. That is,

$$6 + {}^{-}5 = 1.$$

This is read "Six plus negative five equals one."

Continuing with our example of the game, suppose another person scores 10 points on the first play, x points on the second play, and has a total score of 7 points at the end of two plays. That is,

$$10 + x = 7.$$

Again, we are keeping our agreement to always add scores to get the total score. In this case, we see that the person lost 3 points on the second play. However, x is not the number 3, since 10 + 3 = 13, not 7. So again we create a new number, $^{-}3$ (read "negative three"), which added to 10 produces a sum of 7. That is,

$$10 + {}^{-}3 = 7.$$

In our examples of winning and losing points, we have introduced two numbers, $^-5$ and $^-3$. But we are not going to stop with just these two numbers. Instead, we are going to produce a whole new set by creating many new numbers to join to the whole numbers. In this new set every equation of the type "$x + a = b$" will have a solution. This will be progress, since (as we noticed in the last section) we cannot solve every such equation with just the whole numbers alone.

Before trying the exercises, study the following equations and the new numbers which we create as solutions to them. Be sure that you can interpret each of the equations in terms of the game in which you gain or lose points.

Equation	Solution
$5 + x = 1$	$^-4$
$6 + x = 0$	$^-6$
$21 + x = 14$	$^-7$
$5 + x = 8$	3

4.4 Exercises

1. Tell what number is a solution of each of the following equations:

   (a) $17 + x = 3$

   (b) $21 + x = 28$

   (c) $28 + x = 21$

   (d) $105 + x = 83$

   (e) $83 + x = 105$

   (f) $47 + x = 33$

2. How would you describe the equation "$x + 7 = 4$" in terms of a game in which you win or lose points? What must the score have been on the first play?

3. How would you describe the following equations in terms of the game?

$$5 + x = 5$$
$$6 + x = 0.$$

4. Suppose you score 5 points on the first hand, then lose 7 points on the second hand. Then your total score is $5 + {}^{-}7$. What number can be used to show your total score at the end of two hands?

5. Add the following numbers:

   (a) $8 + {}^{-}4$
   (b) $8 + {}^{-}8$
   (c) $8 + {}^{-}12$
   (d) $7 + {}^{-}6$
   (e) $7 + {}^{-}7$
   (f) $7 + {}^{-}8$
   (g) $13 + {}^{-}18$
   (h) $5 + {}^{-}20$
   (i) $5 + {}^{-}25$
   (j) $12 + {}^{-}37$
   (k) $11 + {}^{-}18$
   (l) $126 + {}^{-}315$

6. The temperature at 8 in the morning of a cold winter day is 5 degrees below zero. We should not use the number 5 to show this temperature, since most people would think this meant five above; but it would be reasonable to use ${}^{-}5$ (negative five) to mean five degrees below zero. So, we'll say the 8 o'clock temperature is ${}^{-}5$.

   Now between 8 o'clock and noon, the temperature changes. Suppose x is the number of degrees the temperature changes. Then the temperature at noon is

$${}^{-}5 + x.$$

   With this as a start, try to answer the following questions.

(a) If the temperature rises 5 degrees, what is *x*? What is the noon temperature?

(b) If the temperature rises 7 degrees, what is *x*? What is the noon temperature?

(c) If the temperature *falls* 5 degrees, what is *x*? What is the noon temperature?

(d) Explain why *x* is not the same in parts (a) and (c).

(e) If the temperature does not change at all between 8 o'clock and noon, what is *x*? What is the noon temperature?

7. Add the following. Be sure that you can describe each one in terms of old temperature, temperature change, and new temperature.

(a) $^{-}2 + 3$
(b) $^{-}2 + {}^{-}3$
(c) $^{-}2 + 0$
(d) $5 + 5$
(e) $^{-}5 + {}^{-}5$
(f) $^{-}5 + 5$
(g) $10 + {}^{-}12$
(h) $^{-}10 + 12$
(i) $0 + {}^{-}2$
(j) $^{-}2 + 2$
(k) $^{-}2 + {}^{-}2$
(l) $^{-}1 + 10$
(m) $^{-}15 + 19$
(n) $^{-}15 + 30$
(o) $^{-}15 + 45$

8. A merchant buys a television set for 200 dollars, and he sells it for 200 + x dollars. Answer the following questions.

(a) If he sells the set for 25 dollars more than he paid for it, what is *x*? What is 200 + x ?

(b) If he sells the set for the same price he paid for it, what is *x*? What is 200 + x?

(c) If he sells the set for 25 dollars *less* than he paid for it, what is *x*? What is 200 + x?

(d) Why is x not the same in parts (a) and (c)?

9. A football team has gained 9 yards on the first three downs. If x is the number of yards the team gets on the fourth down, then the total number of yards for the four downs is

$$9 + x.$$

Answer the following questions.

(a) If the team gains 5 yards on the fourth down, what is x? What is $9 + x$?

(b) If the team loses 5 yards on the fourth down, what is x? What is $9 + x$?

(c) Why is x not the same in parts (a) and (b)?

(d) If the team loses 9 yards on the fourth down, what is x? What is $9 + x$?

(e) If the team loses 15 yards on the fourth down, what is x? What is $9 + x$?

10. Add the following:

(a) $37 + 85$
(b) $37 + {}^{-}85$
(c) ${}^{-}37 + 85$
(d) ${}^{-}37 + {}^{-}85$
(e) ${}^{-}102 + 84$
(f) ${}^{-}67 + {}^{-}35$
(g) ${}^{-}18 + 38$
(h) ${}^{-}14 + {}^{-}92$
(i) $14 + 92$
(j) ${}^{-}72 + 12$
(k) $72 + {}^{-}12$
(l) $100 + {}^{-}25$
(m) $100 + {}^{-}25$
(n) ${}^{-}100 + {}^{-}25$
(o) ${}^{-}200 + {}^{-}300$
(p) $200 + {}^{-}300$
(q) ${}^{-}1250 + 250$
(r) ${}^{-}1250 + {}^{-}250$

11. Make up a problem about each of the following situations which uses negative numbers as well as whole numbers:

(a) elevation above and below sea level

(b) gaining and losing weight

(c) increasing and decreasing speed

(d) gains and losses in the stock market

4.5 The Integers and Opposites

The exercises in Section 4.4 suggest that for every whole number (except 0) it is useful to create a new negative number. If we put these numbers together with the whole numbers, we have a new set which may be shown as follows:

$$\{ 0, 1, {}^{-}1, 2, {}^{-}2, 3, {}^{-}3, 4, {}^{-}4, \ldots \}$$

This new set of numbers is called the set of integers, and is referred to as the set Z. The new numbers we have created are called negative integers; for instance, $^{-}1$, $^{-}2$, and $^{-}3$ are negative integers. The other numbers in the set are simply the whole numbers. However, in this new set we shall call the whole numbers (except 0) positive integers; for instance 1, 2, and 3 are positive integers. The number 0 is neither positive nor negative. You may remember that in our illustrations, 0 represented neither a gain nor a loss. So the set Z of integers is made up of the positive integers, the negative integers, and zero.

Suppose that a person scores 5 points on the first play of a game, and x points on the second play. What must x be if his total score at the end of two plays is 0? x must be a number such that

$$5 + x = 0.$$

And since we know that $5 + {}^{-}5 = 0$, we see that x must be -5. It is also true that ${}^{-}5 + 5 = 0$. (What kind of scoring on the two plays does ${}^{-}5 + 5$ show?)

Since 5 and $^{-}5$ add to 0, they are called opposites, or opposite integers. Thus, the opposite of 5 is $^{-}5$, and

the opposite of $^{-}5$ is 5.

In general,

Two integers are opposites if their sum is zero.

What number must be added to 0 in order to produce a sum of 0? Since $0 + 0 = 0$, we shall say that the integer 0 is its own opposite. That is,

The opposite of 0 is 0.

Instead of writing "the opposite of," we shall use the symbol "-" to mean "opposite of." So, "$-2 = {}^{-}2$" may be read as

the opposite of two is negative two.

Then "$-({}^{-}2) = 2$" may be read as

the opposite of negative two is two.

If we use a to stand for an integer, what does "-a" mean? "-a" stands for the opposite of the integer a. Here are some examples:

if $a = 3$, then $-a = {}^{-}3$;

if $a = {}^{-}3$, then $-a = 3$;

if $a = {}^{-}5$, then $-a = 5$;

if $a = 5$, then $-a = {}^{-}5$;

if $a = 0$, then $-a = 0$.

Notice that -a may be a positive integer, a negative integer, or zero.

Questions: What kind of integer is a if -a is positive?

What is a if -a is zero?

What is *a* if -*a* is zero?

We already know that the sum of an integer and its opposite is 0. So, if *a* is an integer, we have

$$\boxed{\begin{aligned} a + (-a) &= 0 \\ -a + a &= 0 \end{aligned}}$$

What kind of meaning can we give to "-(-a)," where *a* is an integer? Let us try to build this expression piece by piece, as follows:

a	*a* is an integer.
$-a$	This is the opposite of the integer *a*.
$-(-a)$	This is the opposite of the integer -*a*.

So "-(-a)" may be read as "the opposite of the opposite of *a*."

Example 1: 3 is an integer.

$-3 = {}^{-}3$ This is the opposite of the integer 3.

$-(-3) = -({}^{-}3)$ This is the opposite of the integer ${}^{-}3$;

$= 3$ that is, 3.

Notice that the opposite of the opposite of 3 is 3! In Example 2, we begin with a negative integer.

Example 2: ${}^{-}5$ is an integer.

$-({}^{-}5) = 5$ This is the opposite of ${}^{-}5$.

$-(-({}^{-}5)) = -5$ And this is the opposite of the opposite of ${}^{-}5$. Note again that the opposite of the opposite of ${}^{-}5$ is ${}^{-}5$. In general, if *a* is an integer, we have

$$\boxed{-(-a) = a}$$

4.6 Exercises

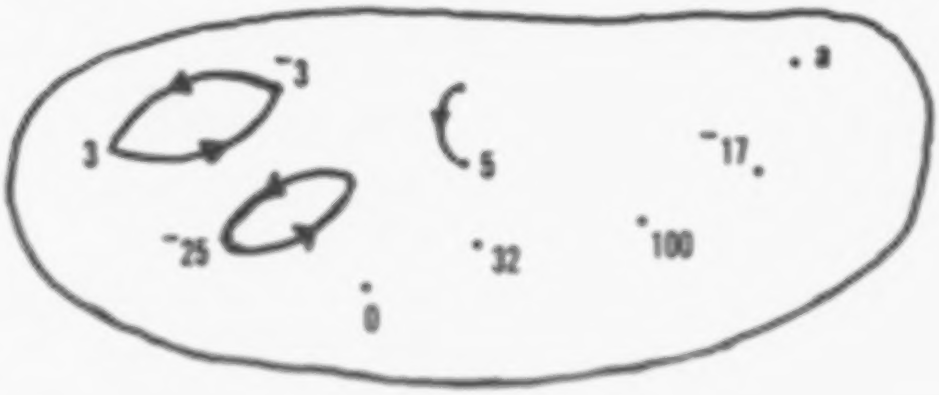
1. Find the solution of each of the following equations.

   (a) $3 + x = 0$ (d) $^{-}72 + b = 0$

   (b) $x + {}^{-}2 = 0$ (e) $^{-}13 + 13 = x$

   (c) $a + 5 = 0$ (f) $0 = 15 + n$

2. Copy and complete the following diagram in which an arrow is to be drawn from each listed integer to its opposite.

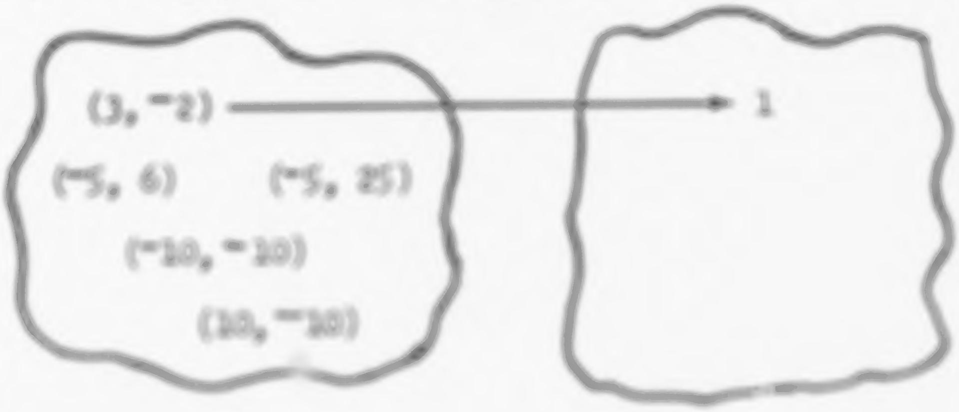


3. If $\underline{x}$ is an integer, $-(-x) =$

4. $-(-(-30)) =$

5. If $\underline{a}$ is an integer, $-(-(-a)) =$

6. Copy and complete the following diagram, showing assignments made by adding pairs of integers.



7. $(a,b) \longrightarrow 4$

   Name five different pairs of integers to which 4 is assigned by addition of integers.

8. $(a,b) \longrightarrow 0$

   Name five different pairs of integers to which 0 is assigned by addition of integers.

9. (a) By using set notation and listing several elements in the set, show the set Z of integers.
   (b) Show a set A having five elements, all of them negative integers.
   (c) Show a set B having five elements, all of them positive integers.
   (d) Show a set C which contains all integers that are neither positive nor negative.

4.7 Properties of (Z,+)

We have seen how physical situations suggest a way in which any two integers may be added. That is, to each ordered pair of integers may be assigned an integer which is their sum. So, addition is a binary operation on the set Z of integers: and (Z,+) is an operational system.

The system (Z,+) has some properties which we have met before. For example, scoring 8 points on the first play of a game and then losing 5 points on the second play gives the same total score as losing 5 points on the first play and winning 8 on the second. That is, $8 + {}^{-}5 = {}^{-}5 + 8$. And in general,

$$\boxed{a + b = b + a}$$

Addition of integers is commutative.

If you score $\underline{a}$ points on the first hand, and then 0 points on the second, the total score remains $\underline{a}$. This suggests the following property:

$$\boxed{a + 0 = 0 + a = a}$$

0 is the identity element of (Z,+).

From Section 4.5, we already know that every integer a has an opposite -a, such that

$$\boxed{a + (-a) = -a + a = 0}$$

For every integer a, -a is its inverse.

You will remember from Chapter 2 that two elements are inverses in a system if they combine to give the identity element.

If a football team gains 5 yards on the first down, loses 3 yards on the second down, and gains 4 yards on the third down, then the total yardage for the three downs may be found in the following way:

$$\begin{aligned}(5 + {}^-3) + 4 &= 2 + 4 \\ &= 6.\end{aligned}$$

We get the same result in the following way:

$$\begin{aligned}5 + ({}^-3 + 4) &= 5 + 1 \\ &= 6.\end{aligned}$$

This example suggest the following property:

$$\boxed{(a + b) + c = a + (b + c)}$$

Addition of integers is associative.

Because addition of integers is associative, we usually omit the parentheses and write simply "a + b + c" to show the sum of three integers.

4.8 Exercises

1. Find the following sums in the way indicated by the parentheses.

(a) $({}^-8 + 7) + {}^-3$ (e) $(16 + {}^-10) + 5$

(b) $({}^-6 + {}^-6) + 9$ (f) $30 + (110 + {}^-50)$

(c) $^-6 + (^-6 + 9)$ (g) $86 + (^-36 + {}^-85)$

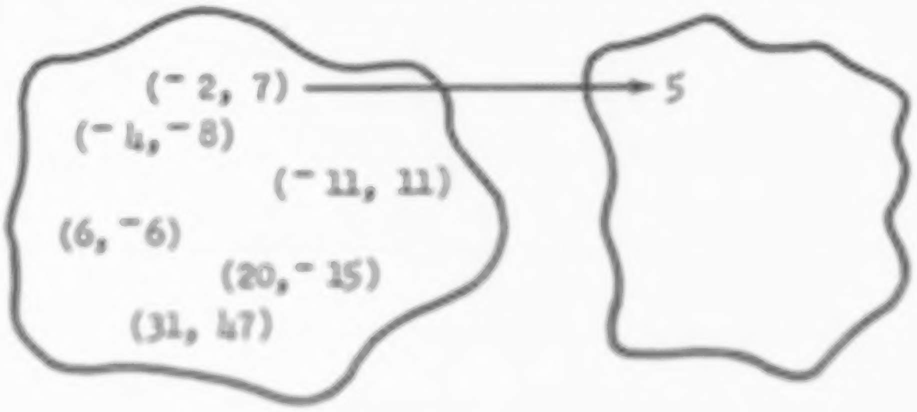
(d) $14 + (^-8 + 5)$ (h) $85 + (^-85 + {}^-36$

(i) $(85 + {}^-5) + -36$

2. Find the following sums.

(a)	$^-8 + 10$	(g)	$18 + 165 + {}^-18$
(b)	$^-8 + 10 + {}^-5$	(h)	$-615 + 108 + 312$
(c)	$15 + {}^-2 + 7$	(i)	$-3 + 5 + {}^-7 + 14$
(d)	$-3 + {}^-5 + {}^-42$	(j)	$8 + {}^-7 + {}^-8 + {}^-7$
(e)	$^-9 + 7 + 18$	(k)	$^-15 + {}^-4 + 6 + {}^-11$
(f)	$42 + {}^-31 + 17$	(l)	$102 + -33 + {}^-25 + 61$

3. Copy and complete the following assignments, illustrating that addition is a binary operation on Z.



4. We have seen that (Z,+) has the following properties:

(i) Associativity

(ii) There is an identity element

(iii) For each element, there is an inverse element

(iv) Commutativity

(a) Tell which of these properties the system $(Z_5, +)$ has.

(b) Tell which of these properties the system (W,+) has.

(c) Review the operational system using rotations of a square (see Exercise 4 in Section 1.12) and tell which of these four properties it has.

(d) Review the definition of a commutative group in Section 2.15. Which of the following operational systems are commutative groups?

$(Z,+)$, $(Z_4,+)$, rotations of a square

4.9 The Integers and Translations on a Line

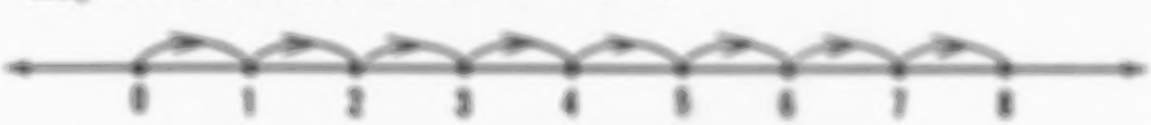
We have already seen that positive and negative integers may be used to represent situations in which "opposites" are involved (winning and losing, rising and falling, forward and backward). Now we shall look at some mappings of the points on a line which the integers may be used to describe.

Below is a number line with some of the whole number points labeled.

In Chapter 3, we saw that the mapping

$$n \longrightarrow n + 1$$

may be illustrated as follows:

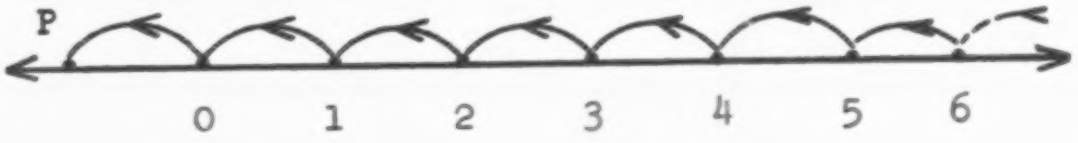


In this mapping, each whole number point is mapped onto a point of the line located one "step" to the right. Also, as a result of this mapping, each whole number n has an image $n + 1$. Notice especially that the image of 0 is 1 under the mapping $n \longrightarrow n + 1$.

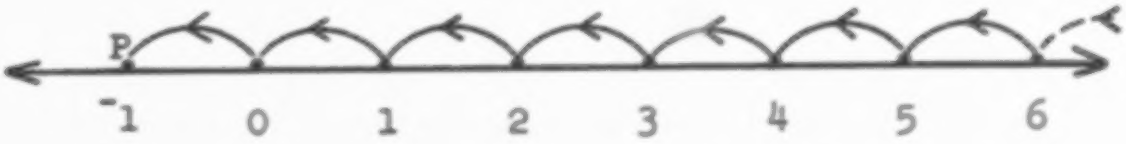
Now what happens to the points of the line under the rule

$$n \longrightarrow n + {}^{-}1?$$

It is reasonable to say that this rule sends the points of the line one step in the direction opposite to that of the mapping $n \longrightarrow n + 1$, since 1 and $^{-}1$ are opposite integers. This may be shown as below.



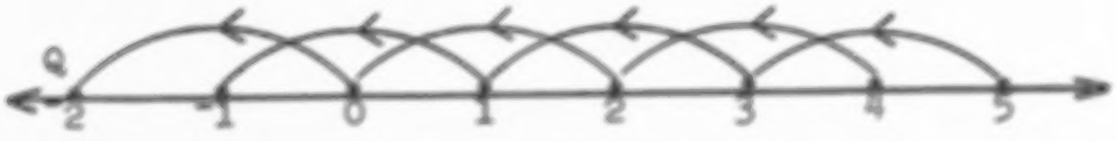
Notice the point marked P in the diagram. So far we have not associated a number with this point, but now it seems reasonable to assign the number $^{-}1$, as below.



In this way, the image of 0 under the rule $n \longrightarrow n + {}^{-}1$ is $^{-}1$.

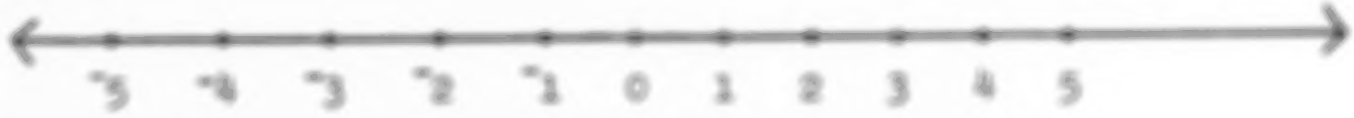
Next, consider the rule

$$n \longrightarrow n + {}^{-}2.$$



The point marked Q is the image of the "zero point" under this rule. So with the point Q we associate the number $^{-}2$.

By using rules such as those above, we may associate every integer -- positive, negative, and zero -- with a point of the line. A part of this number line is shown below.



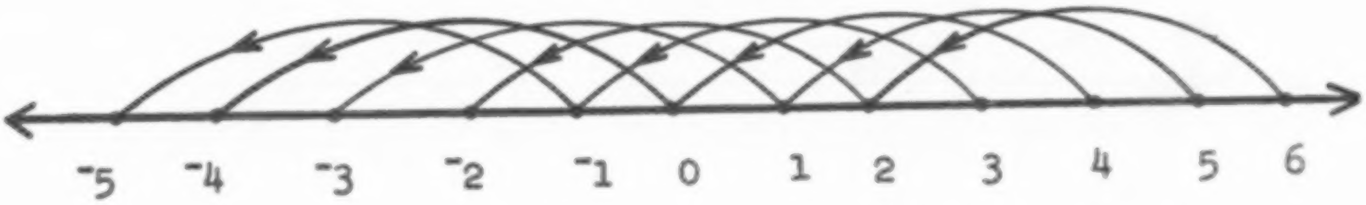
Thus every rule of the kind

$$n \longrightarrow n + a,$$

where a is an integer, is a mapping of the integers into the integers. Furthermore, such a mapping may be used to describe a translation of the line.

Example 1: Describe the translation of the line given by

$$n \longrightarrow n + {}^{-}4.$$



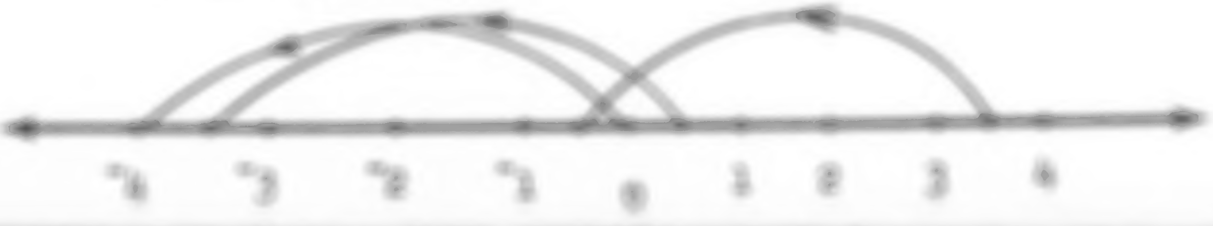
This translation maps every point of the line onto the point 4 steps to the left. Do you see that every point is the image of exactly one point?

Using the integers associated with some of these points, we may say:

the image of 0 is ⁻4;
the image of 10 is 6;
the image of ⁻204 is ⁻208;
3 is the image of 7;
⁻107 is the image of ⁻103.

In fact, in a translation of the line, every point of the line has exactly one image, and every point of the line is the image of exactly one point. Every point has an image, and every point is an image. This includes points other than those associated with the integers, as suggested in the picture below, for the mapping

$$n \longrightarrow n + {}^{-}4$$



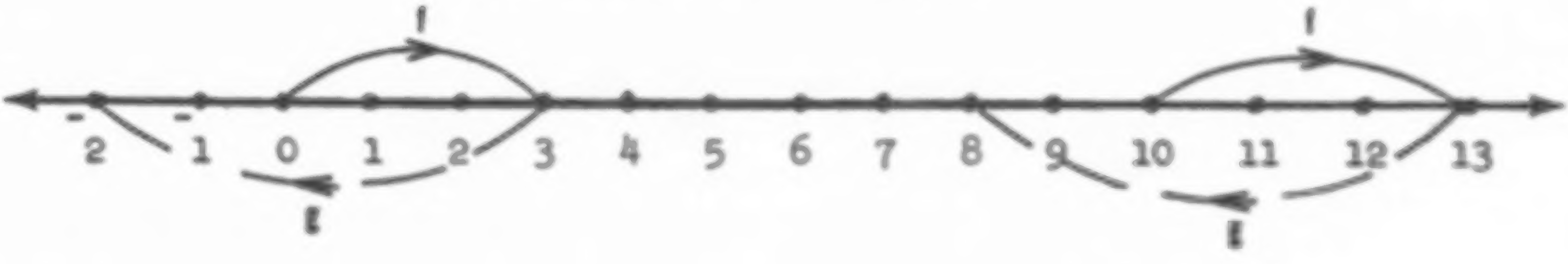
Example 2: Suppose that f is the translation

$$n \longrightarrow n + 3,$$

and g is the translation

$$n \longrightarrow n + {}^{-}5$$

What is the image of 10 under the composition $g \circ f$ (g following f)? First, the translation $n \xrightarrow{f} n + 3$ "shifts" each point of the line three places to the right.

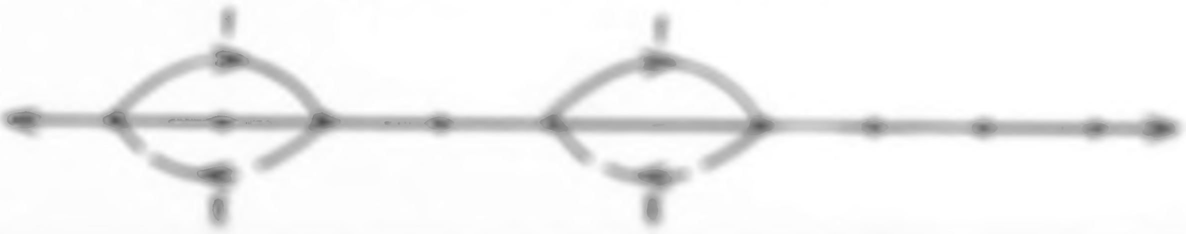


Then g shifts each point five places to the left. So, under this composition of translations the image of 10 is 8. Notice also that the image of 0 is ⁻2.

From Example 2, we can see that the composition $g \circ f$ is the same as the single translation $n \longrightarrow n + {}^{-}2$. This shows us another way to interpret addition of integers, since the sum $3 + {}^{-}5$ is ⁻2.

Questions: If $n \xrightarrow{f} n + {}^{-}3$ and $n \xrightarrow{g} n + {}^{-}7$, what single translation is the same as $g \circ f$? What single translation is the same as $f \circ g$?

Example 3: Suppose $n \xrightarrow{f} n + 2$ and $n \xrightarrow{g} n + {}^{-}2$. The composition $g \circ f$ is illustrated below.



Each point of the line is its own image under this composition. In other words g o f is the identity translation. It is also easy to see that f o g is the identity translation. Therefore,

$$n \longrightarrow n + 2 \text{ and } n \longrightarrow n + {}^-2$$

are inverse translations. (Notice also that $2 + {}^-2 = 0$.)

4.10 Exercises

1. Draw a line, and show the points associated with the following integers: $^-4$, $^-3$, $^-2$, $^-1$, 0, 1, 2, 3, 4. By means of arrows, show the images of these points under the translation $n \xrightarrow{f} n + {}^-3$.
2. Using the translation f and the diagram from Exercise 1, show the composition g o f, where $n \xrightarrow{g} n + 7$.
3. What single translation is the same as g o f in Exercise 2?
4. Draw a line, and use arrows to show the images of points under the translation $n \longrightarrow n + 0$. What is the name of this translation?
5. If $n \xrightarrow{f} n + {}^-7$ and $n \xrightarrow{g} n + 7$, what is f o g? What is g o f?
6. If $n \longrightarrow n + a$, give a description of a translation g so that g o f is the identity translation of a line.
7. For each of the following pairs of translations, tell the single translation which is the same as g o f, and the single translation which is the same as f o g.

	f	g
(a)	$n \longrightarrow n + {}^-7$;	$n \longrightarrow n + {}^-5$
(b)	$n \longrightarrow n + {}^-28$;	$n \longrightarrow n + 15$
(c)	$n \longrightarrow n + 7$;	$n \longrightarrow n + 38$
(d)	$n \longrightarrow n + {}^-5$;	$n \longrightarrow n + 71$
(e)	$n \longrightarrow n + a$;	$n \longrightarrow n + b$
(f)	$n \longrightarrow n + 63$;	$n \longrightarrow n + 0$
(g)	$n \longrightarrow n + 63$;	$n \longrightarrow n + {}^-63$

8. In Exercise 7, is $f \circ g$ the same as $g \circ f$? Is composition of translations commutative?

9. For each of the following, describe a translation g so that $g \circ f$ is the translation $n \longrightarrow n + {}^-27$.

(a) $n \longrightarrow n + {}^-13$

(b) $n \longrightarrow n + 27$

(c) $n \longrightarrow n + {}^-27$

(d) $n \longrightarrow n + 100$

(e) $n \longrightarrow n + {}^-17$

(f) $n \longrightarrow n + a$

10. Suppose the following translations of the line are given:

$$n \xrightarrow{f} n + -8$$

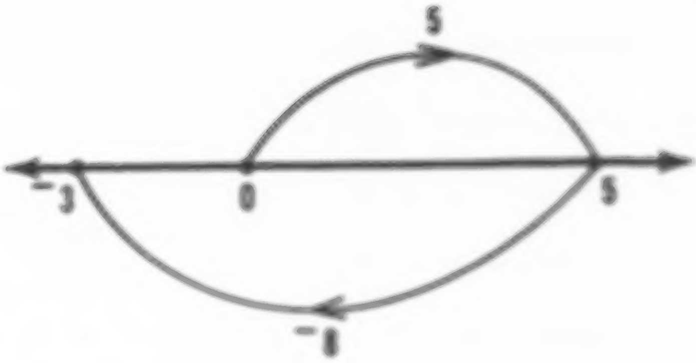
$$n \xrightarrow{g} n + 17$$

$$n \xrightarrow{h} n + {}^-5$$

(a) What single translation is the same as $h \circ (g \circ f)$?

(b) What single translation is the same as $(h \circ g) \circ f$?

11. We have already seen that addition is a binary operation on Z; that is, a sum may be assigned to every ordered pair of integers. Line translations may be used to illustrate addition. As an example, take the sum $5 + {}^-8$.



If we first apply the translation $n \longrightarrow n + 5$, and then follow it with the translation $n \longrightarrow n + (^{-}8)$, the diagram at the left shows that the image of 0 is $^{-}3$.
The composition of these translations $n \longrightarrow n + (^{-}3)$. And this tells us that the sum $5 + (^{-}8) = {}^{-}3$.

Draw diagrams like the one above to illustrate the following sums.

(a) $-2 + -8$ (f) $^{-}18 + 5$
(b) $7 + {}^{-}3$ (g) $18 + {}^{-}5$
(c) $4 + 5$ (h) $^{-}7 + 0$
(d) $10 + {}^{-}10$ (i) $(^{-}2 + 5) + 3$
(e) $-6 + {}^{-}7$ (j) $^{-}2 + (5 + 3)$

*12. Let T be the set of all translations of a line which are of the form

$$n \longrightarrow n + a,$$

where a is an integer.

(a) If two of these translations are applied, one after the other, is the result another such translation?

(b) If "o" is used to mean composition of translations in T, is (T,o) an operational system?

(c) Is composition of these translations associative? (See Exercises 11i and 11j.)

(d) Is composition of these translations commutative?

(e) Is there an identity translation in T?

(f) Does each of the translations in T have an inverse?

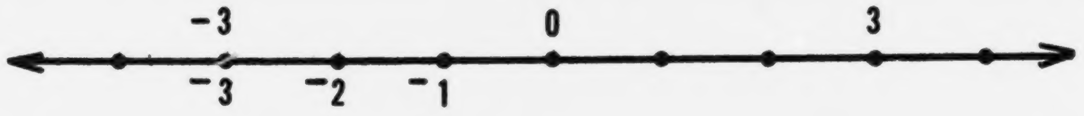
(g) Is the system (T,o) a commutative group?

4.11 Subtraction in (Z,+)

We know that the opposite of 3 is $^{-}3$. That is,

$$-3 = {}^{-}3.$$

so "-3" and "$^{-}3$" are simply names for the same number; they both refer to the same point on the number line:



Since we do not need two different names for the same number, we shall from now on use "-3" to mean not only "opposite of 3" but also "negative 3." In the same way, "-10" may be read as either "opposite of 10" or "negative 10." Be very careful, however, about a symbol such as "-a"; this symbol refers to the opposite of the integer a, which is not necessarily a negative number. (When is -a a positive number?)

So far we have worked only with addition of integers. Is it possible to subtract integers? For example, what if you were asked to subtract -3 from 5; do you know what the difference should be? Let us look at this question carefully.

First, we write

$$5 - (-3)$$

This "-" appears between two numbers, and means to subtract the second number from the first.

This "-" is part of the symbol "-3," a name for negative 3 (or the opposite of 3).

Let us just admit that we do not know what this difference is, and call it a. Then we can write

$$5 - (-3) = a.$$

Now compare this subtraction problem with one which uses only positive integers. For example, we know that

$$5 - 4 = 1.$$

since 1 + 4 = 5. In other words, subtracting 4 from 5 means finding a number to which 4 may be added to produce 5. You may have checked subtraction problems in arithmetic by using this idea:

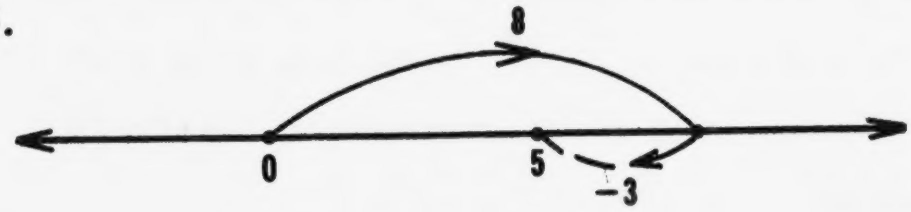
$$\begin{array}{r} 5 \\ -4 \\ \hline 1 \end{array} \qquad\qquad \text{Check} \quad \begin{array}{r} 1 \\ +4 \\ \hline 5 \end{array}$$

This gives us a clue as to how to find the number in our problem, for we would like subtraction to behave the same way in our new set Z of integers as it did with just the whole numbers. So,

$$\text{if } 5 - (-3) = a,$$

$$\text{we want} \quad a + (-3) = 5.$$

In other words, a must be a number to which -3 can be added to product 5. The diagram below should help you to see that this number is 8.



Therefore, 5 - (-3) = 8, since 8 + (-3) = 5.

Let us look at another problem in subtraction of integers.

$$4 - 7 = a$$

The number a must be such that a + 7 = 4. What number can be added to 7 so that the sum is 4? The only such number is -3.

Therefore, we have

$$4 - 7 = {}^{-}3$$

since we already know

$$-3 + 7 = 4.$$

These examples show that you can always subtract integers simply by using your knowledge of addition. Study the following examples.

Example 1: $3 - (-14) = 17$, since $17 + (-14) = 3$

Example 2: $28 - 13 = 15$, since $15 + 13 = 28$

Example 3: $-17 - 5 = -22$, since $-22 + 5 = -17$

Example 4: $-33 - (-15) = -18$ since $-18 + (-15) = -33$

4.12 Exercises

Find the differences.

1. $5 - 2$
2. $5 - (-2)$
3. $-5 - 2$
4. $-5 - (-2)$
5. $10 - 7$
6. $7 - 10$
7. $13 - 5$
8. $5 - 13$
9. $-3 - (-2)$
10. $-3 - 2$
11. $-15 - (-8)$
12. $100 - (-100)$
13. $100 - 100$
14. $100 - 200$
15. $100 - (-200)$

4.13 Subtraction as Addition of Opposites

Our work with subtraction has suggested that we can always subtract by adding. And in fact there is an important mathematical principle which shows this relationship. For example, $5 - (-3) = 8$, as we saw earlier. But notice that instead of subtracting -3 we can add the opposite of -3; that is $5 + (3) = 8$. In other words,

$$5 \underset{\text{subtracting an integer}}{-} (-3) \underset{\text{is the same as}}{=} 5 + \underset{\text{adding its opposite}}{(3)}.$$

To look at another case, we know that $-7 - (2) = -9$. However, instead of subtracting 2, we may add its opposite;

$$-7 + (-2) = -9.$$

$$-7 - 2 = -7 + (-2) = -9$$

In this way, we may express every subtraction problem as an addition problem. Instead of subtracting a number, we may add its opposite. We state this in the following way:

$$\boxed{a - b = a + (-b).}$$

Use this principle in exercises which follow.

4.14 Exercises

1. $80 - (-20)$
2. $-25 - 75$
3. $-25 - (-75)$
4. $14 - 7$
5. $14 - (-7)$
6. $-14 - (-7)$

11. $-167 - 82$
12. $55 - (-55)$
13. $55 - 55$
14. $1,681,352 - (-2,684,917)$
15. $-3,066,502 - (-8,300,070)$
16. $a - (-2) =$

7.	$-14 - 7$	17.	$-2 - (-a) =$
8.	$-87 - 95$	18.	$-a - (-b) =$
9.	$-87 - (-95)$	19.	$x - (-7) =$
10.	$167 - 82$	20.	$a - [-(b + c)] =$

21. Is subtraction an operation on the set Z of integers? Is (Z,-) an operational system? Explain your answer. Is subtraction of integers commutative? Is it associative?

Consider the expression

$$-5 + 2 - 4 + 7.$$

In this expression, the "-" between "2" and "4" means to subtract 4. And in an expression involving additions and subtractions, we agree to perform the operations in order from left to right. Hence we have

$$\begin{aligned} -5 + 2 - 4 + 7 &= -3 - 4 + 7 \\ &= -7 + 7 \\ &= 0. \end{aligned}$$

Also we may rewrite the original expression as one involving addition:

$$-5 + 2 + (-4) + 7,$$

since we have seen that adding the opposite of 4 is the same as subtracting 4. Since addition is both commutative and associative, we may take the numbers in any order. For example,

$$\begin{aligned} -5 + 2 + (-4) + 7 &= [(-5) + (-4)] + [2 + 7] \\ &= -9 + 9 \\ &= 0. \end{aligned}$$

In Exercises 22-35, rewrite each expression as one involving only addition. Then simplify.

22. $7 - 8 + 2$

23. $-14 + 5 - (-3)$

24. $42 - (-2) - 7$

25. $216 - 38 - (-10)$

26. $-316 - 55$

27. $-316 + 55$

28. $8 + 7 - 15 - 32$

29. $-28 - 32 + 17 - (-3)$

30. $15 - 7 + 8 - 3 + 4$

31. $23 + 31 - 45 + 51 - 87$

32. $18 - 19 + 25 - 72 + 33 - 80$

33. $3 + 5 - 3 - 5$

34. $-7 - 2 + 7 + 2$

35. $-8 + 3 + 8 - 3$

What is $5 + 2 - 5 - 2$? It is the same as

$$5 + 2 + (-5) + (-2).$$

And since addition of integers is both associative and commutative, we may think of this as

$$(5 + -5) + (2 + -2),$$

which is $0 + 0$, or 0. In other words, we have

$$(5 + 2) + (-5 + -2) = 0.$$

Therefore, we know that $(-5 + -2)$ is the opposite of $(5 + 2)$, since the sum is zero. In symbols, we may write this as

$$-(5 + 2) = (-5) + (-2).$$

Now let a and b be any two integers. What is $-(a + b)$? as above, we see that

$$a + b + (-a) + (-b) = 0.$$

Therefore, the opposite of a + b is (-a) + (-b). In symbols, we have

$$\boxed{-(a + b) = (-a) + (-b).}$$

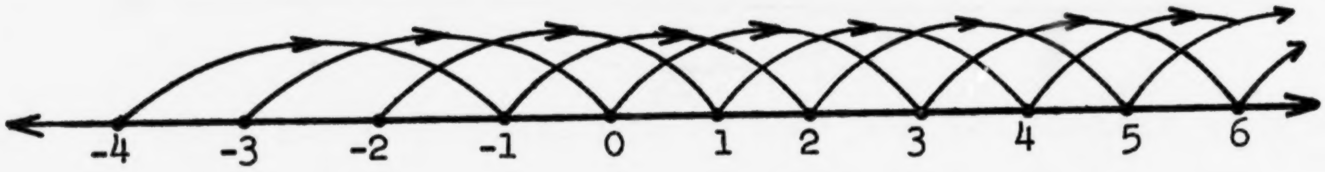
Use this principle in Exercises 36-45. All letters represent integers.

36. $-(x + y)$
37. $-(-x + y)$
38. $-[-x + (-y)]$
39. $-(7 + a)$
40. $-(a - 4)$
41. $-(a - b)$
42. $-(a + b + c)$
43. $-(a + b - c)$
44. $-(a - b - c)$
45. $-[-(x + y)]$

4.15 Equations in (Z,+)

Below is a diagram illustrating the line translation

$$n \xrightarrow{f} n + 3.$$

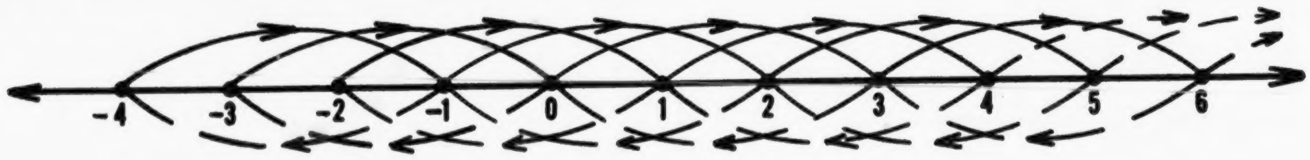


We studied such translations in Section 4.9, and we saw at that time that each of them has an inverse translation. The inverse of the translation

$$n \xrightarrow{f} n + 3$$

is $n \xrightarrow{g} n - 3$. (This of course is the same as $n \longrightarrow n + (-3)$.)

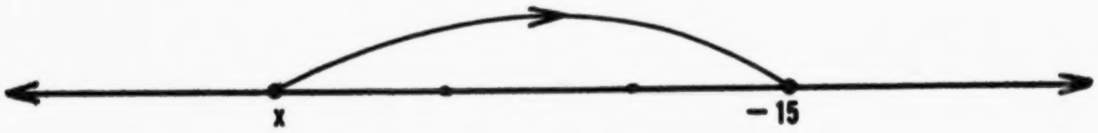
We know that the composition g o f is the identity translation; this is illustrated below.



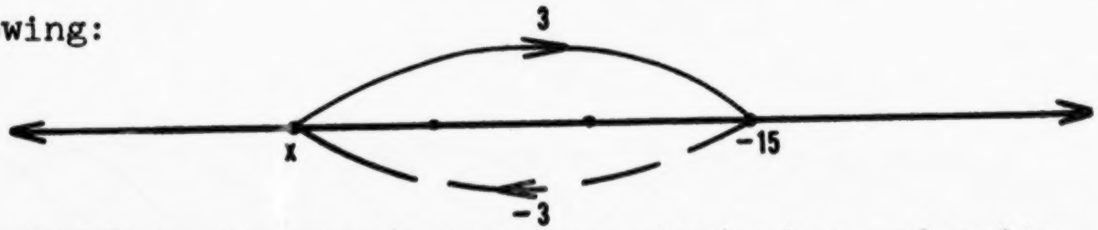
Now look at the equation

$$x + 3 = -15.$$

Does this equation have a solution in the set Z of integers? You see below that we may picture this equation in terms of the line translation $n \xrightarrow{f} n + 3$.



In other words, x must be a number whose image under f is -15. Now if we follow $n \longrightarrow n + 3$ by its inverse $n \longrightarrow n - 3$ we have the following:



From this diagram, we see that we may start at x and write

$$x + 3 = -15$$

or we may start at -15 and write

$$-15 - 3 = x.$$

In other words, any x which is a solution of one of these equations is a solution of the other also. Therefore, to solve the original equation we may proceed as follows:

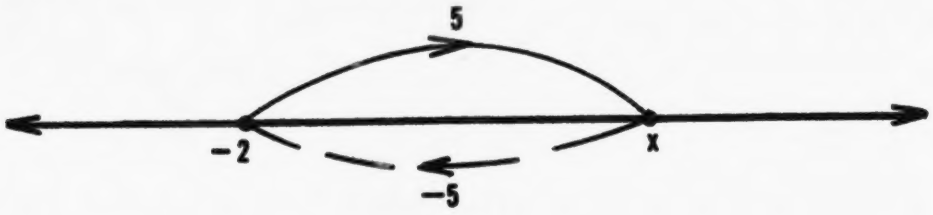
$$x + 3 = -15$$
$$x = -15 - 3$$
$$x = -18$$

since $-15 - 3$ is the same as $-15 + (-3)$. This is easy to check, because we know

$$-18 + 3 = -15.$$

Below are two more examples of solving equations of the type "$x + a = b$," where a and b are integers.

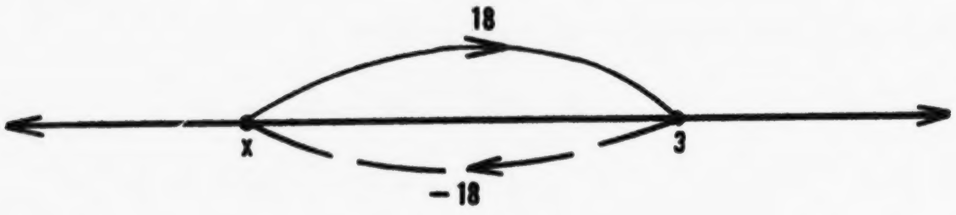
Example 1: $x - 5 = -2$



$$-2 + 5 = x$$
$$3 = x$$

$\{3\}$ is the solution set.

Example 2: $x + 18 = 3$



$$3 - 18 = x$$
$$-15 = x$$

$\{-15\}$ is the solution set.

4.16 Exercises

Solve the following equations in $(Z,+)$.

1. $x + 3 = 1$
2. $x + 3 = 3$

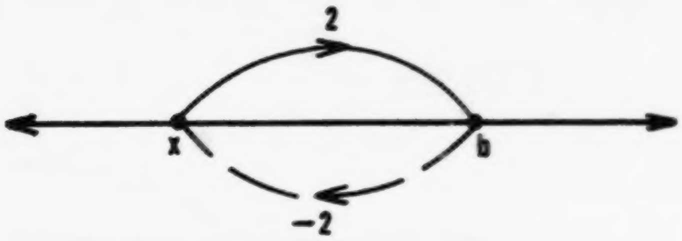
11. $a + (-5) = 8$
12. $a - 5 = 8$

3. $3 + x = 1$
4. $3 + x = 3$
5. $x + 2 = 7$
6. $x + 8 = -2$
7. $x + (-2) = 8$
8. $x - 2 = 8$
9. $a + 4 = -4$
10. $4 + a = -4$
13. $a - 27 = 33$
14. $a - 27 = -4$
15. $x - 15 = -2$
16. $x - 15 = 0$
17. $x - 15 = -3$
18. $x + 15 = -3$
19. $3 + a = -100$
20. $3 - a = -100$

Look at the equation

$$x + 2 = b.$$

Can you solve it for *x*?



We see that if $x + 2 = b$, then $b - 2 = x$. Also if $b - 2 = x$, then $x + 2 = b$. So to solve the equation "$x + 2 = b$" we write

$$x + 2 = b$$

$$x = b - 2.$$

Now work Exercises 21-30, solving for *x*.

21. $x + 3 = b$
22. $x + 5 = b$
23. $x + 100 = b$
24. $x + (-6) = b$
25. $x - 6 = b$
26. $x + 15 = b$
27. $x + (-10) = b$
28. $x - 10 = b$
29. $x - 14 = b$
30. $x + a = b$
31. Use Exercise 30 to answer the following question: Does every equation of the type "$x + a = b$," where *a* and *b* are integers, have a solution *x* in the set of integers?

32. Make up ten different equations of the type

$$x + a = b,$$

where a and b are integers, and solve the equations.

4.17 Cancellation Law

Suppose that a and b are integers, and that

$$5 + a = 5 + b$$

What conclusion can you draw? If your conclusion is that a and b are actually the same integer (that is, $a = b$), then you are using a cancellation law. (You may want to review Section 2.11 if you do not remember what a cancellation law is.) Is it correct to use a cancellation law in the case above? We start by knowing that

$$5 + a = 5 + b.$$

The inverse of the integer 5 is -5, and we may write

$$-5 + (5 + a) = -5 + (5 + b).$$

Do you see that on both sides of "=," we have the same sum? We know that $5 + a$ is the same as $5 + b$, and certainly -5 is the same as -5. Now we may write

$$(-5 + 5) + a = (-5 + 5) + b,$$

since addition of integers is associative. Then

$$0 + a = 0 + b,$$

since the sum of an integer and its opposite is 0. Do you see now why we chose the integer -5 at the beginning? Since 0 is the identity element of (Z,+), we may finally write

$$a = b.$$

So, if $5 + a = 5 + b$, then $a = b$; and we see that our use of a cancellation law in this case was correct. Can we always use a cancellation law in such a case? That is, is there a cancellation law in (Z,+)? Study the following steps, where a, b, and c are integers.

IF $c + a = c + b$

THEN

$-c + (c + a) = -c + (c + b)$ Why?

THEN

$(-c + c) + a = (-c + c) + b$ Why?

THEN

$0 + a = 0 + b$ Why?

THEN

$a = b$ Why?

You should be able to answer each of the questions "Why?" since the argument here is the same as the earlier one, except in this case c represents any integer. So we may write, where a, b, and c are integers,

IF	$c + a = c + b$
THEN	$a = b$

Cancellation Law of (Z,+)

Example: Use the cancellation law of (Z,+) to solve the equation

$$-3 + x = -1.$$

First, we may rewrite the equation as

$$-3 + x = -3 + 2,$$

since $-3 + 2 = -1$. We now have -3 on both sides of "=," and we may use the cancellation law to get

$$x = 2.$$

Therefore we have another way in which to solve equations in (Z,+) besides the method we used in Section 4.15.

4.18 Exercises

1. May the cancellation law of (Z,+) be stated in the following way: If $a + c = b + c$, then $a = b$?

2. (a) Give an example of an operational system besides (Z,+) in which there is a cancellation law.

   (b) Give an example of an operational system in which there is no cancellation law.

3. Use the cancellation law of (Z,+) to solve the following equations.

   (a) $2 + x = 2 + (-5)$
   (b) $2 + x = 2 - 5$
   (c) $x + (-7) = -3 + (-7)$
   (d) $x - 7 = -3 - 7$
   (e) $n - 5 = -10 -5$
   (f) $y + 43 = -14 + 43$
   (g) $-2 + t = -2 + 19$
   (h) $-2 + t = -2 - 19$

4. Use the cancellation law of (Z,+) to solve the following equations.

   (a) $5 + x = 17$
   (b) $-4 + x = 12$
   (c) $x + (-3) = -6$
   (d) $x - 3 = -6$
   (e) $-13 + x = 42$
   (f) $19 + x = -13$
   (g) $3 + n = 47$
   (h) $n - 10 = -5$
   (i) $y - 14 = 7$
   (j) $-32 + x = 32$
   (k) $x + 3 = 7$
   (l) $x + a = b$ (Solve for x.)

4.19 Ordering the Integers

It is important to recognize that numbers can be compared in the sense of saying that one number is less than another. For instance, 3 is less than 7 and we write

$$3 < 7.$$

We may also show the comparison of these numbers by saying that "7 is greater than 3" and writing

$$7 > 3.$$

3 and 7 of course are positive integers, but it should be possible to compare any two integers, positive, negative, or zero. For example, which of the two integers, 3 and -7 is greater? Is -10 less than -3 or greater than -3? After studying this section, you should be able to answer questions such as these.

We think of 7 as being greater than 3 because we must add a positive integer to 3 to get 7. Specifically, $3 + 4 = 7$; 4 is the number we add to 3 to get 7. Of course, saying $3 + 4 = 7$ is the same as saying $7 - 3 = 4$. Thus, $7 > 3$, and the difference $7 - 3$ is the positive integer 4. In the same way, $10 > 8$, and the difference $10 - 8$ is the positive integer 2.

We should like to keep this same pattern in comparing any two integers. Therefore, we shall say that if a and b are integers

a - b is a positive integer
means
$a > b$, and
$b < a$.

With this agreement, let's return to the two questions we asked earlier. Which is greater, 3 or -7?

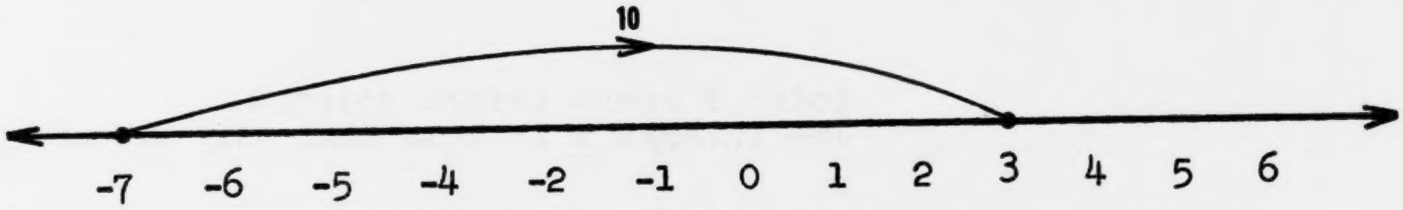
-7 + -3 = -10. -10 is a negative integer; so -7 is not greater than -3.

3 - (-7) = 10. 10 is a positive integer; so 3 is greater than -7.

We may write 3 > -7

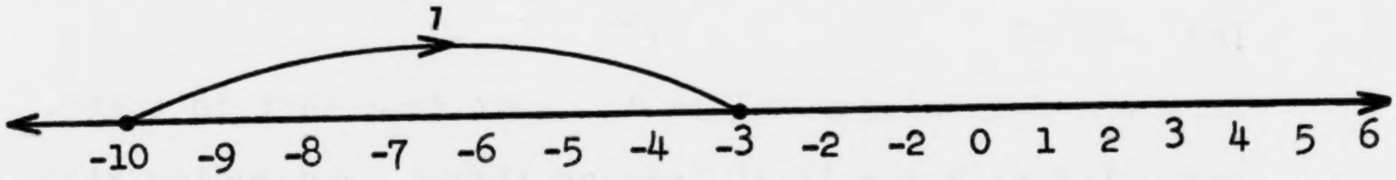
-7 < 3.

The number line may also be used to show that the difference 3 - (-7) is 10.



3 - (-7) = 10 means that 10 + (-7) = 3, or by commutativity, -7 + 10 = 3, as shown above. You must add 10 (or shift to the right 10 steps) to get 3. This means that the point associated with -7 is to the left of the point associated with 3.

Which is greater, -10 or -3? The number line below illustrates that -10 + 7 = -3.

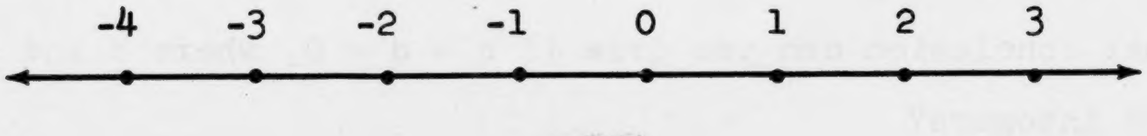


In other words, -3 - (-10) = 7, a positive integer. Therefore,

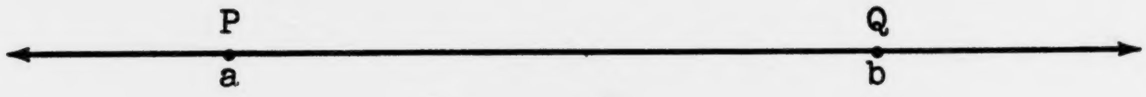
-3 > -10, and

-10 < -3.

By looking at the number line below, can you tell at a glance which of the two numbers, -3 and 2, is greater?



Having decided on a way to order the integers, we see that we can also use this ordering to order the points of the number line to which integers have been assigned. In this way, given two different points, the one to the left is said to come before the one to the right. And the integer assigned to the left point is less than the integer assigned to the right point.



Point P comes before point Q.
The integer a is less than the integer b.

4.20 Exercises

1. For each of the following pairs of integers, tell which is greater and why.

(a) -6, 2
(b) 6, -2
(c) -6, -2
(d) 0, -1
(e) 0, 1
(f) -6, -7

2. List the following integers in order from left to right, beginning with the least integer listed, and ending with the greatest integer listed.

2, -2, 3, -5, 0, -1, 4, -4, -3, 5, 1.

3. If a - b is a positive integer, which is the greater integer, a or b?

If r - s is a negative integer, which is the greater integer, r or s?

What conclusion can you draw if c - d = 0, where c and d are integers?

4. Copy the following pairs of integers, and insert the symbol "=," "<," or ">," whichever makes a true sentence.

(a) 7 -3

(b) -5 -15

(c) -8 0

(d) 8 0

(e) -100 2

(f) 1 -2500

(g) 5 14

(h) -5 -14

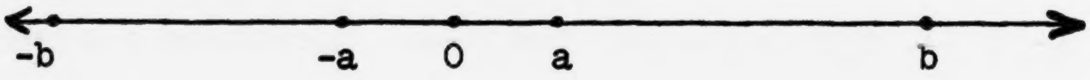
(i) 7 15

(j) -7 -15

(k) -3 5

(l) 3 -5

5. Look at the number line diagram below. Which integer a, or b, is greater?



Can you complete the following sentence: If $a < b$, then

$-a$ __________.

6. For each of the following pairs of integers, tell which is greater:

(a) 7, -7

(b) -5, 5

(c) -13, 13

(d) 52, -52

(e) 33, -33

(f) -97, 97

7. (a) If x is a negative integer, and y is a positive integer, which is greater, x or y?

(b) If x is a negative integer, and $y = 0$, which is greater, x or y?

(c) If x is a positive integer, and $y = 0$, which is greater, x or y?

8. Suppose that a, b, and c are three integers, and you know the following:

$$a < b, \text{ and } b < c.$$

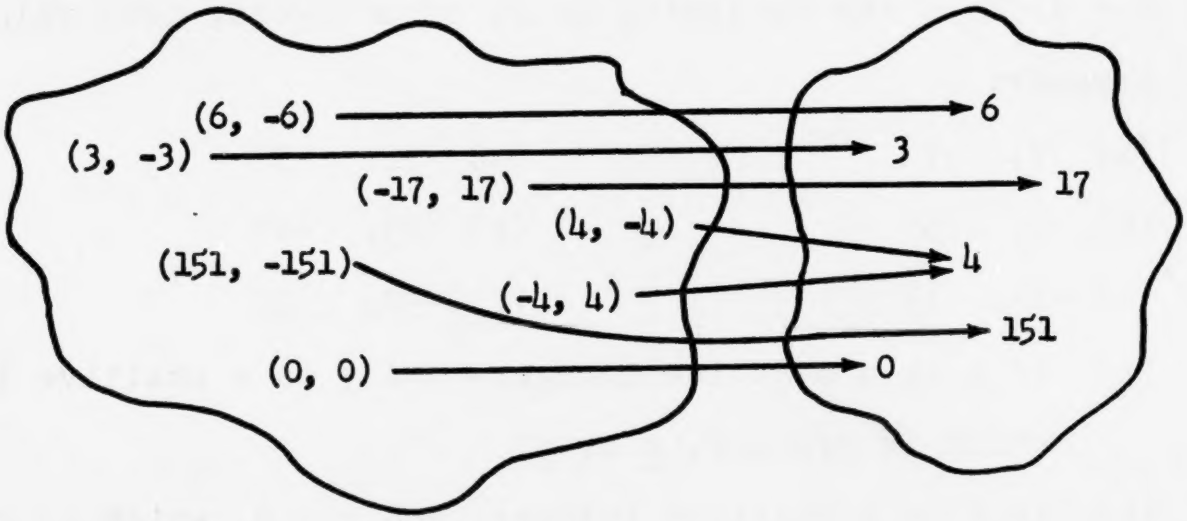
What other fact do you know? Illustrate on a number line.

4.21 Absolute Value

In Chapter 2 we studied an operation on the whole numbers called the "max" operation. For example,

$$(2,6) \longrightarrow 6, \text{ or } 2 \max 6 = 6,$$

since 6 is the greater number of the pair. Instead of "2 max 6 = 6," we may also write "max(2,6) = 6"; the meaning is the same. "Max" is also an operation on the integers. Given the pair (-3,-7), for instance, the "max" operation assigns the number -3, since $-3 > -7$. In the diagram below, illustrating the "max" operation, we use only a special kind of pair; each pair consists of an integer and its opposite.



Each of these pairs (except (0,0)) is assigned a positive integer. Why? Of the two numbers in each pair, one is positive and the other negative; and the positive number is the greater.

Example 1: Suppose a is a positive integer.
Then $\max(a,-a) = a$, since a is the greater integer of the pair.

Example 2: Suppose a is a negative number.
Then $\max(a,-a) = -a$. In this case, since a is a negative number, $-a$ is a positive number. And the positive number is greater than the negative number.

It is often useful in mathematics to work with the number $\max(a,-a)$ where a is an integer; and this number is called the *absolute value* of a.

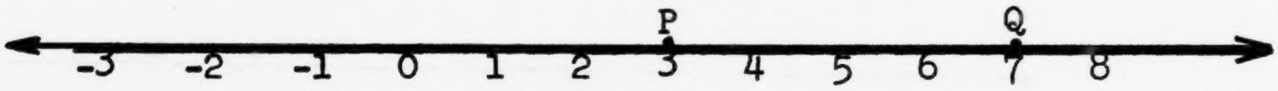
Example 3. What is the absolute value of -10? The opposite of -10 is 10. And $\max(-10,10) = 10$. Therefore, the absolute value of -10 is 10.

Instead of writing the words "absolute value of," we shall simply use the symbol "$|a|$" to mean "absolute value of a."

Example 4: $|-7| = \max(-7,7) = 7$.

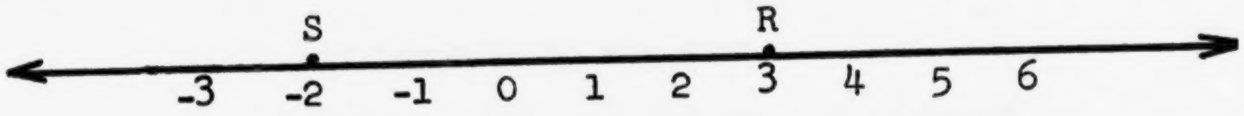
Example 5: $|25| = \max(25,-25) = 25$

On the number line below, it is reasonable to say that



the *distance* between points P and Q is 4, since it takes 4 "steps" to get from one point to the other. We could find this distance by subtraction $7 - 3 = 4$, where 7 and 3 are the integers associated with the points P and Q. Notice that if we subtract in the other order, $3 - 7$, we get -4. We do not use this for the distance between two different points.

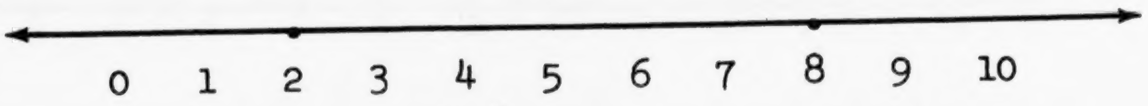
Example 6. What is the distance between R and S?



Here too the distance can be found by subtracting the numbers associated with the points, $3 - (-2) = 5$. Again notice that if we subtract in the other order, we get $-2 - 3$, or -5. But the distance is the positive number 5.

Now suppose that we want to find the distance between two points, point W which is associated with the integer a on the number line, and point Y, which is associated with the integer b. Is the distance $a - b$ or $b - a$? We cannot be sure in this case; one of these numbers is positive, and one is negative. However, if we take the absolute value, we are sure to get a positive number, regardless of which of the two numbers we choose. So it is correct to say the distance is $|a - b|$; it is also correct to say the distance is $|b - a|$.

Example 7: What is the distance between the points shown below?



$$|8 - 2| = |6|, \text{ or } |2 - 8| = |-6| = 6$$

We may subtract in either order as long as we use the absolute value for the distance.

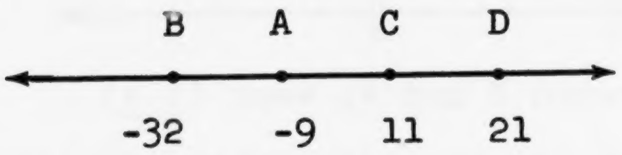
4.22 Exercises

1.
(a) max (7, -7) =
(b) |-7| =
(c) |215| =
(d) |-215| =
(e) max (-215,215) =
(f) |3| =
(g) |-83| =
(h) max (-83,83) =
(i) |100| =
(j) |-100| =
(k) |max (-3,-6)| =
(l) max (-3, |-6|) =

2. Since we have previously agreed that 0 is its own opposite, what is |0|?
(Remember that max (a,a) = a.)

3. Find the simplest name for each of the following:
(a) |3 - 7|
(b) |7 - 3|
(c) |100 - 18|
(d) |18 - 100||
(e) |5 - (-2)|
(f) |-2 - 5|
(g) |-7 - 14|
(h) |-14 - (-7)|
(i) |62 - 37|
(j) |37 - 62|
(k) |10 - (-38)|
(l) |-38 - (-10)|

4. Using the number line below, find the distance



between the following points:
(a) B and C
(b) C and D
(c) B and A
(d) B and D
(e) A and C

5. Find the simplest name for each of the following:

   (a) $|7 - 0|$

   (b) $|0 - 7|$

   (c) $|-7 - 0|$

   (d) $|0 - (-7)|$

   (e) $|-18 - 0|$

   (f) $|0 - (-18)|$

   (g) $|18 - 0|$

   (h) $|0 - 18|$

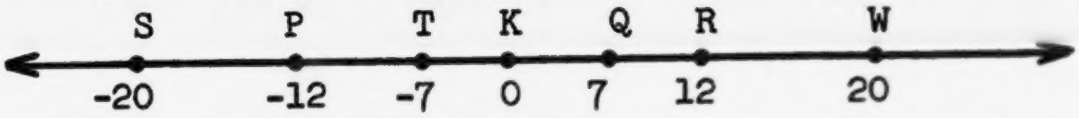
   (i) $|100 - 0|$

   (j) $|0 - 100|$

   (k) $|-100 - 0|$

   (l) $|0 - (-100)|$

6. Using the number line below, find the distance



between the following points:

   (a) T and K

   (b) Q and K

   (c) P and K

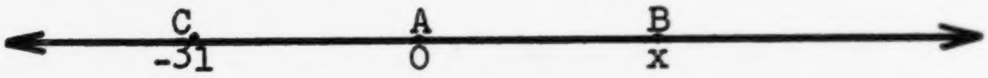
   (d) R and K

   (e) S and K

   (f) W and K

   (g) P and W

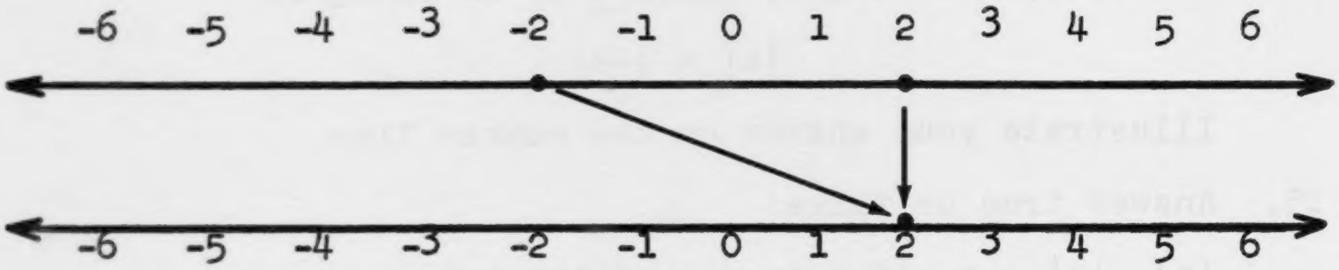
7. In the diagram below, if the distance between C and A is the



same as the distance between B and A, what is $\underline{x}$?

8. On a number line, point O is associated with the integer 0, and point P is associated with the integer $\underline{p}$. What is the distance between O and P?

9. Complete the following drawing, in which an arrow is drawn from each of the given integers to its absolute value.



10. Which of the following is not true for any integer a?

$|a| > 0$; $|a| = 0$; $|a| < 0$.

11. Find solution sets for the following equations:

(a) $|a| = 5$

(b) $|a| = 0$

(c) $|a| = -5$

( ) $|a| = 100$

(e) $|x| + 1 = 9$

(f) $|x + 1| = 9$

(g) $|n + 2| = 3$

(h) $|n| + 2 = 3$

(i) $|a - 1| = 5$

(j) $|a| - 1 = 5$

(k) $|n + 7| = -2$

(l) $|n + 7| = 0$

12. Describe the integers which are solutions of the following: (It may help to use the number line.)

(a) $|a| < 2$

(b) $|a| > 2$

(c) $|a| < 5$

(d) $|a| > 5$

(e) $|a| < 0$

(f) $|a| > 0$

(g) $|a| < 100$

(h) $|a| > 100$

*13. Describe the integers which are solutions of the following:

(a) $|x + 2| < 2$

(b) $|x - 2| < 2$

(c) $|x - 3| < 7$

(d) $|x + 3| < 7$

14. Answer true or false, where a is an integer:

$$|a| = |-a|$$

Illustrate your answer on the number line.

15. Answer true or false:

(a) $|a| = a$, if a is a positive integer.

(b) $|a| = -a$, if a is a negative integer.

(c) $|a| = a$, if a is 0.

*16. (a) $6 + 2 = 8$. This example illustrates that the sum of two positive integers is positive. What is $|6|$? What is $|2|$? What is $|6| + |2|$? What is $|8|$? Notice that the absolute value of the sum is the same as the sum of the absolute values. Make up a rule for finding $a + b$, where both a and b are positive integers.

(b) $-6 + (-2) = -8$. This example illustrates that the sum of two negative integers is negative. What is $|-6|$? What is $|-2|$? What is $|-6| + |-2|$? What is $|-8|$? Notice that the sum of the absolute values is the same as the absolute value of the sum.

Make up a rule for finding $a + b$, where both a and b are negative integers.

(c) $6 + (-2) = 4$. Here we are adding a positive integer and a negative integer. Notice that $|6| + |-2| = 6 + 2 = 8$, and this is not the same as $|4|$. In other words, the sum of the absolute values in this case is not the same as the absolute value of the sum.

Which has the greater absolute value, 6 or -2? Notice that the sum, 4, is a positive integer, just as 6 is (and, of the two numbers being added, 6 has the greater absolute value). What is $|6| - |-2|$? Make up a rule for finding a + b, where _a_ is a positive integer, _b_ is a negative integer, and $|a| > |b|$.

(d) $-6 + 2 = -4$. Here we are adding a positive integer and a negative integer, and the sum is negative. Which has the greater absolute value, -6 or 2? Notice that the sum, -4, is a negative integer just as -6 is (and of the two numbers being added, -6 has the greater absolute value). What is $|-6| - |2|$? Make up a rule for finding a + b, where _a_ is a negative integer, _b_ is a positive integer, and $|a| > |b|$.

*17. When is it true that $|a + b| = |a| + |b|$? (Be sure to consider cases in which either _a_ or _b_, or both, are zero.)

4.23 Summary

1. The set Z of _integers_ is made up of the positive integers, zero, and the negative integers.
2. Every integer _a_ has an _opposite_, -_a_, such that $a + (-a) = 0$. If _a_ is positive, -_a_ is negative. If _a_ is negative, -_a_ is positive. If _a_ is zero, -_a_ is zero.
3. $-(-a) = a$.
4. The _absolute value_ of an integer _a_ is written as "$|a|$." $|a| = \max(a,-a)$. Therefore, $|a|$ is never negative.

5. The distance on the number line between the points associated with the integers *a* and *b* is $|a - b|$. This distance is also $|b - a|$, since $|a - b| = |b - a|$.

6. *Addition* is an operation on the set Z of integers. That is, to every ordered pair of integers is assigned an integer called their sum. Therefore, (Z,+) is an *operational system*.

7. The operational system (Z,+) has the following properties:
   (i) Associativity
   (ii) Commutativity
   (iii) Identity element
   (iv) Inverse element for each element.

   Therefore, (Z,+) is a *commutative group*.

8. There is a cancellation law in (Z,+). If $c + a = c + b$, then $a = b$.

9. The integers may be used in many kinds of problems in which the idea of "opposites" occurs. Also the integers may be used to describe certain *translations on a line*, such translations being denoted by $n \longrightarrow n + a$, where *a* is an integer. A line translation is a mapping, since every point of the line is the image of exactly one point.

10. *Subtraction* is an operation on the integers. However, it is not associative and it is not commutative.

11. $a - b = a + (-b)$. Every subtraction may be expressed as an addition.

12. The opposite of a sum is the sum of the opposites.

$$-(a + b) = (-a) + (-b).$$

13. The integers are ordered. $a < b$ if and only if $b - a$ is a positive number. All negative numbers are less than zero; all positive numbers are greater than zero.

14. Every equation of the kind "$x + a = b$," where a and b are integers, has a solution in the set of integers.

4.24 Exercises

In Exercises 1-10, find the sums.

1. $-18 + (-15)$
2. $-34 + (-83)$
3. $32 + (-19) + 58$
4. $107 + 89 + (-16)$
5. $-217 + 88 + (-365) + 47$
6. $-18 + 52 + (-43) + 108 + (-92)$
7. $195 + (-195) + 208 + (-208) + 66$
8. $1257 + (-13335)$
9. $251 + 375 + (-801) + 455$
10. $5681 + 4355 + (-11652)$

In Exercises 11-22, find the differences.

11. $32 - (-8)$
12. $-55 - 17$
13. $-82 - (-19)$
14. $17 - 38$
15. $17 - (-38)$
16. $-45 - 110$
17. $-187 - (-258)$

18. $-258 - 312$
19. $-47 - 85$
20. $-47 - (-85)$
21. $0 - 15$
22. $0 - (-15)$
23. $15 - 8 + 7 - 22 + 13 =$
24. $- 15 + 8 - 7 + 22 - 13 =$
25. $106 + 42 - 38 + 15 - 62 =$
26. $52 + 18 - 93 + 106 - 84 =$
27. $-124 - 35 + 87 - 78 + 39 =$
28. $168 - 3835 + 2106 =$
29. $9857 - 3462 - 2118 =$
30. $12385 - 14689 + 5206 =$
31. Write the following integers in order from left to right, beginning with the least integer listed and ending with the greatest integer listed:

    72, -3, -109, 3, 0, -42, 68, -10, -88, 215, -1000.
32. Between each of the following, insert "<," ">," or "=" whichever results in a true sentence.

    (a) $|-3| \quad -3$

    (b) $|7| \quad 7$

    (c) $(-2 + 8) \quad (-2 - 8)$

    (d) $|3 - 7| \quad |7 - 3|$

    (e) $(0 - 18) \quad (0 - (-18))$

    (f) $|-10| + |-3| \quad |-10 + (-3)|$

    (g) $(42 + (-18)) \quad (42 - (-18))$

    (h) $|7 + (-2)| \quad (|7| + |-2|)$

    (i) $(a + (-a)) \quad (b + (-b))$

    (j) $0 \quad |a + (-a)|$
33. For each of the following, draw a number line. By using arrows, show the translation of the line which it describes.

    (a) $n \longrightarrow n + 5$

    (b) $n \longrightarrow n - 7$

    (c) $n \longrightarrow n + (-3)$

    (d) $n \longrightarrow n + 0.$

34. For each of the following, draw a number line. By using arrows, show the composition g o f of the given translations.

(a) $n \xrightarrow{f} n + 6$; $n \xrightarrow{g} n - 4$.

(b) $n \xrightarrow{f} n + 10$; $n \xrightarrow{g} n - 10$.

(c) $n \xrightarrow{f} n - 9$; $n \xrightarrow{g} n + 5$.

(d) $n \xrightarrow{f} n - 4$; $n \xrightarrow{g} n - 3$.

(e) $n \xrightarrow{f} n + 4$; $n \xrightarrow{g} n + 3$.

35. Tell what single line translation is the same as each of the compositions in Exercise 34.

36. Find the solution set of each of the following equations. (Solve for $\underline{x}$.)

(a) $x + 3 = 7$ (g) $x - 81 = 106$

(b) $x + 7 = 3$ (h) $x - 106 = 81$

(c) $x - 3 = 7$ (i) $x + 7 = b$

(d) $x - 7 = 3$ (j) $x + a = 13$

(e) $x + 81 = 106$ (k) $x + a = b$

(f) $x + 106 = 81$ (l) $x + t = r$

37. Find the solution set of each of the following. (Solve for $\underline{x}$.)

(a) $|x| = 5$ (e) $|x + 2| = 7$

(b) $|x| = -5$ (f) $|x| = x$

(c) $|x| = 0$ (g) $|x| = -x$

(d) $|x| + 2 = 7$ (h) $|x| = |-x|$

38. Tell what integers are solutions of the following sentences:

(a) $|x| < 15$ (c) $|x| > -2$

(b) $|x| > 15$ (d) $|x| < -2$

*(e) $|x - 2| < 5$

*(f) $|x + 5| < 2$

39. Simplify the following:

    (a) $-(a + b)$

    (b) $-(a - b)$

    (c) $-(-a + b)$

    (d) $-(-a - b)$

    (e) $-(x - y + z)$

40. The figure below can be used as an addition table for the integers if properly completed and extended.

								8							
-2			1					6							
								4				8			
					0				4						
						0		2	3		5				
							0	1	2	3					
-8			-5		-3	-2	-1	0	1	2	3	4			7
								-1	0	1	2	3			
								-2	-1	0					
			-8					-3	-2		0				
-13								-5							
						-10		-8							

(a) Copy the table and fill in all the entries.

(b) What do you notice about all the cells having the same number (for example, all cells in which "4" is entered)?

(c) In what way does the table show that addition of integers is commutative?

(d) In what way does the table show that every integer has exactly one inverse for addition?

(e) Try to find at least one other pattern which shows up in the table.

CHAPTER 5

PROBABILITY AND STATISTICS

5.1 Introduction

The Fish and Game Commission often must estimate the number of fish in a lake. But they certainly cannot catch all the fish in the lake and count them. Instead, they catch a sample in a net, tag them, and throw them back into the lake. After allowing time for the first sample to mix thoroughly with the fish population, they catch a second sample and count the number of tagged fish in this sample. The fraction of tagged fish in the second sample is an estimate or a guess of the fraction of tagged fish in the lake. For example, if the first sample numbers 100 and the second sample 200, of which 50 are tagged, it is assumed that about $\frac{50}{200}$ or $\frac{1}{4}$ of the fish in the lake are tagged. Only 100 fish were tagged, so 100 is about $\frac{1}{4}$ of the fish in the lake.

Question: On the basis of the above estimate, how many fish are in this lake?

A similar estimation problem is often met in industry. For instance, in the manufacture of light bulbs it is important to control the equality of the bulbs coming off the assembly line. Since it is not practical to test the burning time of each bulb, a sample of several bulbs is selected and tested. The fraction of defective bulbs in the sample is then used as an estimate of the fraction of defective bulbs in the lot of bulbs being produced. This fraction is called the relative frequency of defective bulbs. If the sample consisted of 50 bulbs, of which 5 were defective then $\frac{5}{50}$ or .1 is the relative frequency of defective bulbs in the sample.

Today many users of mathematics need the ability to make estimates with a high degree of confidence, in situations where the actual results are uncertain. Important decisions are often based on these estimates.

Question: What are some ways relative frequencies might be used by

(a) the weather bureau;
(b) an auto insurance company;
(c) the National Safety Council;
(d) a life insurance company;
(e) the manager of a supermarket?

5.2 Discussion of an Experiment

The experiment that we discuss here is that of tossing a die. You may think of the experiment as a set of trials and an associated set of outcomes. In this case a trial consists of one toss of the die. The possible outcomes are pictured below:

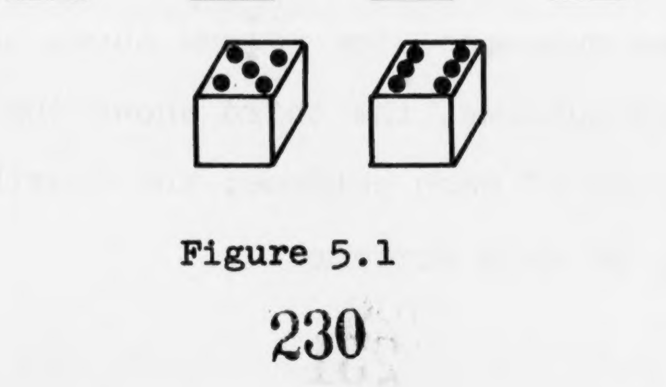


Figure 5.1

We say that the outcome is 2 if the die comes to rest with two dots on the "up" face. The outcome set is {1,2,3,4,5,6}. For each trial the outcome is the number of dots on the up face. If a trial results in a certain outcome, we say that this outcome occurs.

The set of outcomes {1,2,3,4,5,6} is also called the outcome space. Any subset of the outcome space is called an event. Thus, {2,4,6} can be described as the event that the outcome is an even number. An event is said to occur if any one of its outcomes occurs.

We can simplify the description of event {2,4,6} by letting {2,4,6} = K. Then if an outcome is an even number, we say that K occurred. For example, if the outcome of a trial was 2 we say that K occurred.

Since any subset of an outcome set is an event, the outcome set {1,2,3,4,5,6} is itself an event. It could be described as the event that the outcome was a whole number between zero and seven. A subset containing a single outcome is called a simple event, or a point in the outcome space. For example, in this experiment {2} is a simple event or a point.

Below is a table showing the results of an experiment that was performed. The experiment consisted of rolling a die 24 times with the outcome set {1,2,3,4,5,6.} The first column of the table shows the outcomes, the second shows the tally of the occurrences of each outcome; the third shows the frequency of number of occurrences of each outcome; the fourth shows the relative frequency of each outcome.

Table 1 — 24 Tosses of a Die

Outcomes	Tally	Frequency	Relative Frequency
1	///	3	$\frac{3}{24} = \frac{1}{8}$
2	𝍸	5	$\frac{5}{24}$
3	////	4	$\frac{4}{24} = \frac{1}{6}$
4	//	2	$\frac{2}{24} = \frac{1}{12}$
5	𝍸	5	$\frac{5}{24}$
6	𝍸	5	$\frac{5}{24}$

5.3 Exercises

1. Tabulate the following events of the die tossing experiment. That is, list all outcomes that satisfy the condition.
   - (a) The outcome is less than 3. Ans. {1,2}
   - (b) The outcome is greater than 5.
   - (c) The outcome is less than 3 or greater than 5. Where "or" is used, tabulate all outcomes that satisfy at least one of the two conditions.
   - (d) The outcome is greater than 1 and less than 4. Where "and" is used, tabulate only outcomes that satisfy both conditions.
   - (e) The outcome is greater than 2 and less than 3.

(f) The outcome is a member of the outcome set.

(g) The outcome is a prime number.

2. Describe the following tabulated events.

(a) {2,4,6}　　(c) {1,6}

(b) {1,3,5}　　(d) {2,3,5}

3. Perform the experiment of tossing a die 24 times and record the results in a table like Table 1. Compare your results with those in Table 1.

   (a) Which of your frequencies were the same as those in Table 1?

   (b) Which of your relative frequencies were the same?

   (c) Do you think that you will always get the same relative frequencies in repeating this experiment? If you have doubts, try it!

   (d) Add the relative frequencies in the last column of your table. Add the relative frequencies in Table 1. Were the sums the same?

   (e) Find out what the other students in your class found as the sum of the relative frequencies in their tables.

   (f) If you all found the same sum, try to explain why this happened.

4. Suppose that you had a coin with a "head" on both sides and performed the experiment of tossing this coin 100 times.

   (a) What would be the frequency of the outcome "heads-up?"

   (b) What would be the relative frequency of this outcome?

   (c) Would you say that the outcome, heads, was certain?

   (d) What is the relative frequency of any event that is certain?

5. In the same experiment of tossing the "two-headed" coin:
   (a) What is the frequency of the outcome "tails?"
   (b) What is the relative frequency of this outcome?
   (c) If "heads" was a certain event for this experiment, how would you describe the event, "tails?"
6. What is the relative frequency of an event that is impossible?
7. In the die tossing experiment, what is the relative frequency of 2? of 4? of 6?
8. What is the sum of the relative frequencies in Exercise 7?
9. In the die tossing experiment, what is the relative frequency of the event that the outcome is an even number?
10. What conjecture might you make on the basis of the answers to Exercises 7, 8, and 9?
11. Class Discussion Exercise:

    It is interesting to find out what happens to relative frequencies as you increase the number of trials. Instead of repeating an experiment many times, you may save time by combining your results with those of the other students in the class.

    Use the results for the event {5} in your die-tossing experiment (Exercise 3) for the following experiment. First, draw this chart on the chalkboard:

Cumulative Number of Trials	Cumulative Frequency	Relative Frequency
24		
48		
72		
96		
etc.		

(a) Have one student go to the board and enter his frequency and relative frequency for the outcome 5 in the columns to the right of 24.

(b) Have another student go to the board, add his frequency for the same event to the frequency of the first studen and enter the sum in the second row of the cumulative frequency column. Then divide this sum by 48, and enter the quotient (in fraction-form) in the relative frequency column.

(c) Have a third student follow the same procedure in the third row and so on.

(d) If the first three students had 4,3 and 5 respectively, for the frequency of the outcome {5}, entries would look like this:

24	4	$\frac{4}{24}$ or $\frac{1}{6}$
48	7	$\frac{7}{48}$
72	12	$\frac{12}{72}$ or $\frac{1}{6}$

Question: If you have 20 students in your class, how many trials will you have by the time each student has recorded his results on the chart?

(e) Experience indicates that as you increase the number of trials in an experiment to very large numbers, the relative frequencies of an event tend to vary less and less from some specific number. Even though your class project does not involve very large numbers, compute the differences of consecutive pairs of relative frequencies to see if they tend to decrease. (See the illustration below for a suggestion on how to proceed.)

Number of trials	Relative Frequencies	Consecutive Differences
24	a	
48	b	$\|a - b\|$
72	c	$\|b - c\|$
96	d	$\|c - d\|$
etc.	etc.	etc.

(f) The property discussed in this exercise is called the stability of relative frequencies.

(g) The following statements summarize the ideas of the preceding exercises.

1. The relative frequency of an event is
   (a) 0, 1 or a number between 0 and 1;
   (b) 1 if the event is certain;
   (c) 0 if the event is impossible;

(d) the sum of the relative frequencies of its simple outcomes.

2. The sum of the relative frequencies of the outcomes in an experiment is 1.
3. Relative frequency has the property of stability. This idea will be explored further in experiments and illustrated graphically.

5.4 An Experiment to be Performed by Students

For this experiment students should work in groups of two or three, but each one should perform the trials while his teammates help him tally the results. In this way you can do experiments where you need a large number of trials but want to use the same experimental object such as the same die, coin or thumbtack. The large number of trials can be achieved by combining the results of the three members on a team.

(1) Experiment: 20 tosses of a thumbtack repeated 5 times. Toss a thumbtack on a hard surface where it will bounce before coming to rest. (We hope that this will take all of the prejudice out of the way you toss.) The simple outcomes will be "Up" and "Down."

Each student should make a chart like the one below and tabulate his results. When each student on a team has completed 5 groups of 20 tosses, the three students working together should fill in Table 2 for the cumulative results, using the outcome UP. (Is information lost by only considering the outcome UP? Why or why not?) We illustrate how this might begin.

(2) Tables

Twenty Tosses of A Thumbtack Repeated Five Times

	UP			DOWN		
Trials	Tally	Frequency	Relative Frequency	Tally	Frequency	Relative Frequency
1st 20	𝍸 𝍸 //	12	$\frac{12}{20} = \frac{3}{5}$	𝍸 ///	8	$\frac{8}{20} = \frac{2}{5}$
2nd 20	𝍸 ////	9	$\frac{9}{20}$	𝍸 𝍸 /	11	$\frac{11}{20}$
3rd 20						
4th 20						
5th 20						

Table 2

Cumulative Results for 300 Tosses of a Thumbtack

(groups of twenty)

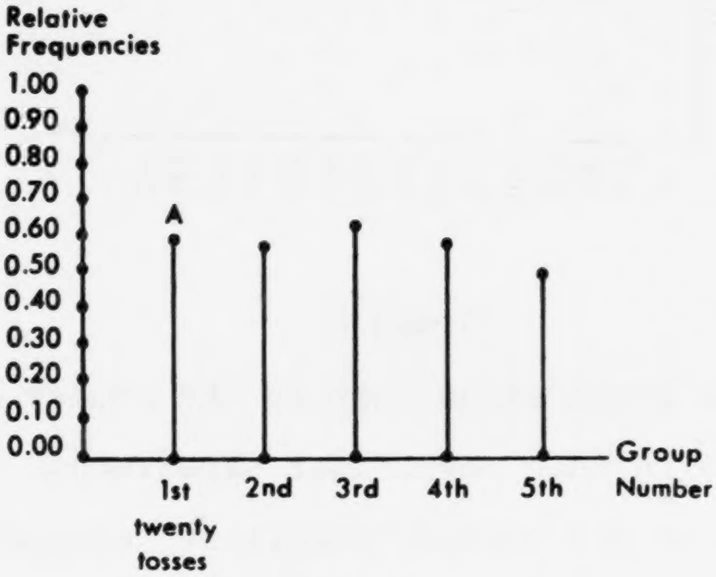
Cumulative Number of Trials	Cumulative Frequency for UP	Relative Frequency for UP	Consecutive Differences
20	12	$\frac{12}{20} = \frac{3}{5} = .6$	
40	21	$\frac{21}{40} = .53$	$\frac{3}{40}$
60	27	$\frac{27}{60} = .45$	$\frac{9}{120}$
80	etc.		
100			
120			
140			
160			
180			
200			
220			
240			
260			
280			
300			

Table 3

(3) Graphs

The best way to illustrate the stability of relative frequency is through the use of graphs, Each student will make two graphs to show the results of the thumbtack experiment. The relative frequencies for UP, tabulated in Tables 2 and 3 will be used.

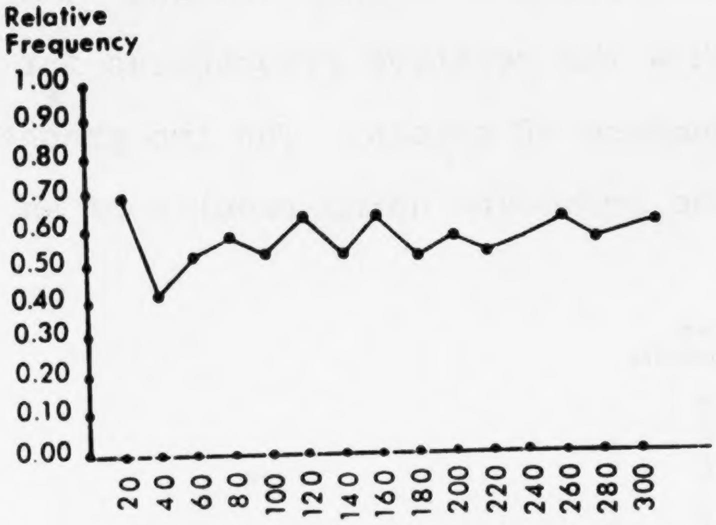
The first graph will show the relative frequencies for UP for each group of twenty tosses. The second graph will show the relative frequencies for UP for increasing numbers of trials. The two graphs below illustrate the procedure using results of an imaginary experiment.



Graph 1

This graph shows that for the five groups of tosses the relative frequency did not vary much. Point A shows that in the first group of twenty tosses the relative frequency of UP was .6. For the five groups illustrated in the graph, the greatest relative frequency was about 0.62 and the least about 0.48. The difference between the greatest and least is 0.14.

Now construct a graph similar to that in Graph 1 using the results of your experiment tabulated in Table 2.



Graph 2

This graph shows that as the number of tosses increased in this particular experiment, the relative frequencies did indeed "stabilize" around a number (about 0.62). With a different thumbtack, the number might have been different. Now construct a graph similar to Graph 2 using the results of your team tabulation in Table 3. Do your relative frequencies tend to stabilize around a number? Is this number near .62? If not, can you explain the difference? Compare your cumulative relative frequency with those of other teams.

Question: What do you think the results might be with a thumbtack that has a very small head and a long pin?

5.5 The Probability of an Event

The thumbtack experiment provided us with one example of the tendency of relative frequencies to "stabilize" as the number of trials increases. This tendency is sometimes called the law of large numbers. This law can be verified by many types of experiments.

Our findings about the stability of relative frequencies suggest that we might be able to predict relative frequencies in some cases where they can't be observed or where it would be very impractical to observe them, For example, if you were manufacturing firecrackers, you wouldn't want to test the quality of your product by exploding each one. (Or maybe you would!) The prediction of relative frequencies is an assignment of numbers to events. The number is called the probability of the event. If you like shorthand, you may use the symbol "P(E)" to stand for the "probability of the event E."

All rights involve responsibilities, and the right to assign probabilities to events obligates us to obey certain laws. Suppose you feel, on the basis of experience, that one of your coins will come up heads about $\frac{1}{3}$ of the time. You decide to assign $\frac{1}{3}$ to P(H) (the probability of heads). What must you then assign to P(T)? In other words, about how often would you expect tails?

In short, since probabilities are predictions of relative frequencies we must expect them to obey all of the properties that we have developed for relative frequencies. Thus P(E), the probability of event E, must satisfy the following:

1. $0 \leq P(E) \leq 1$.
2. $P(E) = 1$, if E is certain to occur.
3. $P(E) = 0$, if E cannot occur.
4. The sum of the probabilities of the outcomes in an outcome set is 1.
5. $P(E)$ is equal to the sum of the probabilities of the simple outcomes in E.

5.6 A Game of Chance

Play the following game with another student in your class and decide if it is fair or unfair. Toss a pair of dice (or wooden cubes with numerals from "1" to "6" on the faces if anyone objects to dice), and observe the sum of the outcomes.

Player A gets one point if the sum is 2,3,4,10,11 or 12. Player B gets one point if the sum is 5,6,7,8 or 9. Notice that there are 6 sums that will give player A a point and only 5 sums that will give player B a point. The first person to get 10 points wins the game.

(a) Pick a partner and play the game 4 times.

(b) How often did player A win? player B?

(c) Is the game fair? If not, who had the advantage?

(d) If one player has the advantage try to discover why.

5.7 Equally Probable Outcomes

You have seen that we can assign probabilities to the simple outcomes of an experiment on the basis of experience with relative frequencies. But even without such experience, it is often reasonable to assign the same probability to each of the simple outcomes in an experiment. For example, in tossing a coin, we often assign equal probabilities to heads and tails.

Question: In this case, what is the probability of heads? tails?

In tossing a die we often assign the same probabilities to 1,2,3,4,5,and 6.

Question: In this case what is the probability of each outcome?

Question: If there are n equally probable outcomes in an outcome set, what is the probability of each?

If we say that a coin or a die is fair, we mean that each element in the outcome set has the same probability.

If we toss a fair die, what is the probability of the event that the outcome is greater than 4? In this event $\{5,6\}$ there are 2 simple events, $\{5\}$, $\{6\}$. $P(\{5\}) = \frac{1}{6}$ and $P(\{6\}) = \frac{1}{6}$. Since the probability of an event is the sum of the probabilities of its simple outcomes, $P(\{5,6\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

When the selection of a member from a set is made so that each possible choice is equally likely, we say that we are selecting a member at random. Consider the experiment of selecting a letter of the alphabet at random. Each letter is equally likely to be chosen. Let V be the event that a vowel is selected, C the event that a consonant is selected, and A the event that a letter in the alphabet is selected.

Questions: What is P(V)? What is P(C)?
What is P(A)?
What is P(V) + P(C)?

5.8 Exercises

1. Toss a pair of dice of different color, for example one white and the other blue. The outcomes occur in ordered pairs (W,B). There are 6 outcomes for the white die and for each of these there are 6 outcomes for the blue die.

Question: How many ordered pairs of outcomes are there for the two dice? Use the order (white,blue). You could record the outcome set in a square pattern as follows:

(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

(a) Copy the above diagram of the outcome space.

(b) Use a ruler to draw a red line through all of the pairs for which the sum is 7. Do the same for sums of 5,6,8 and 9.

(c) Now draw a green line through all the pairs for which the sum is 10. Do the same for sums of 2,3,4,11 and 12.

(d) Let each outcome in the diagram represent a point. How many points are on green lines?

(e) How many points are on red lines?

(f) How many points are in the total outcome set?

(g) If you select a point at random what is the probability that it will be a green line? a red line?

(h) What is the probability that when you toss a pair of dice, the sum of the outcomes for each die will be 5,6,7,8 or 9? 2,3,4,10,11 or 12?

(i) Now look back at the dice game of Section 5.6. Was it a fair game?

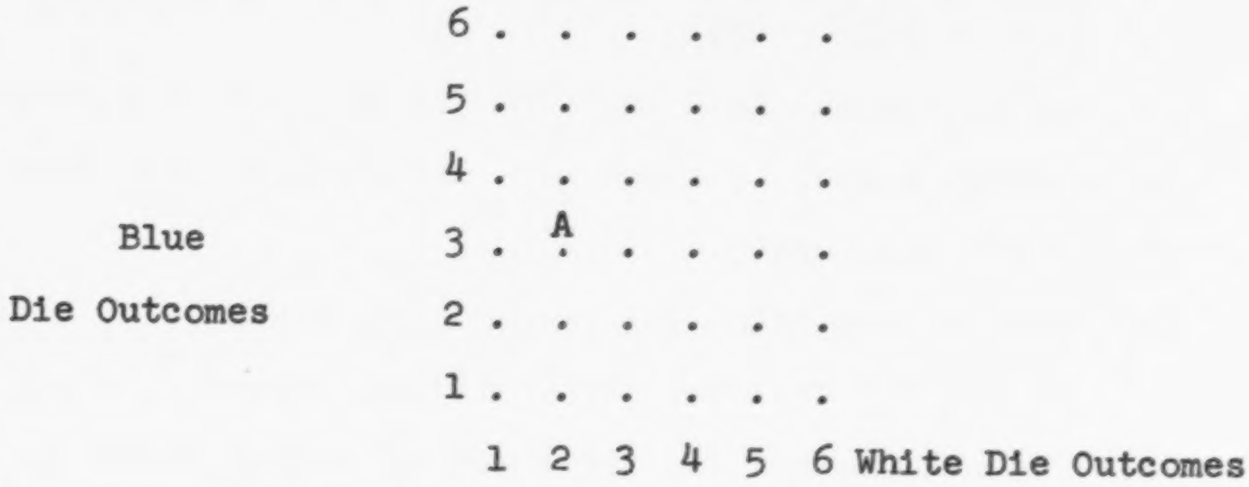
2. (a) Using your diagram from Exercise 1 draw a closed curve around the set of points for the event "The white die outcome is less than the blue die outcome" and call this event "A."

(b) Repeat the directions in (a) for event "The white die outcome is greater than the blue die outcome," and call this event "B."

(c) Let C be the event that "A occurs or B occurs."

(d) What is P(A)? P(B)? P(A) + P(B)? note that P(A) + P(B) = P(C) and that A and B have no outcomes (or points) in common.

3. Make another diagram of the outcome set but this time, to simplify matters, use dots for the points as below:



Point A in the diagram is associated with (2,3). To avoid confusion between single outcomes and pairs of outcomes we will call 2 the first coordinate of point A and 3 the second coordinate of point A.

(a) Draw a line through the points with equal first and second coordinates. Call the set of points on this line "event R."

(b) Draw a line through the points with coordinate sum 8. Call the set of points on this line "event T."

(c) Do R and T have a point in common? If so, what are the coordinates of this point? Call the set with only this common point "event Q."

(d) What is P(R)? What is P(T)? What is P(Q)?

(e) Let K stand for the event "R occurs or T occurs." What is P(K)? What is P(R) + P(T)? Is P(K) = P(R) + P(T)?

(f) Does P(K) = P(R) + P(T) - P(Q)?

(g) Compare the results of this exercise with Exercise 2 and try to discover why in Exercise 2, P(A) + P(B) = P(C) and in Exercise 3, P(K) = P(R) + P(T) - P(Q).

4. It is a well-known fact that the probability of a newborn child being a girl is about $\frac{1}{2}$. What probability does this leave for boys?

(a) What do you think the probability might be of a family having Boy-Girl-Boy (BGB) in that order?

(b) The outcome set for the event of having three children is

{BBB,BBG,BGB,BGG, GBB,GBG, GGB, GGG}.

(c) How many ordered triplets are in the above outcome set?

(d) Assuming all outcomes to be equally probable, what is the probability of each?

(e) How many of the triples tabulated in (b) have G as the second letter?

(f) What is the probability that the second child is a girl?

(g) Suppose that we change our outcome set to include only those outcomes where we know that the first child was a boy.

{BBB, BBG, BGB, BGG}

How many outcomes are in this set?

(h) What is the probability, using the outcome set of (g) that the second child is a girl?

(i) In questions (f) and (h), the answers should be the same. In other words, the fact that the first child was a boy did not influence the likelihood that the second was a girl.

5.9 Another Kind of Mapping

In Chapter 3 you studied mappings from one set of numbers onto another set of numbers and mappings from one set of points onto another set of points. Below is a diagram that portrays a mapping from a set of outcomes onto a set of numbers:

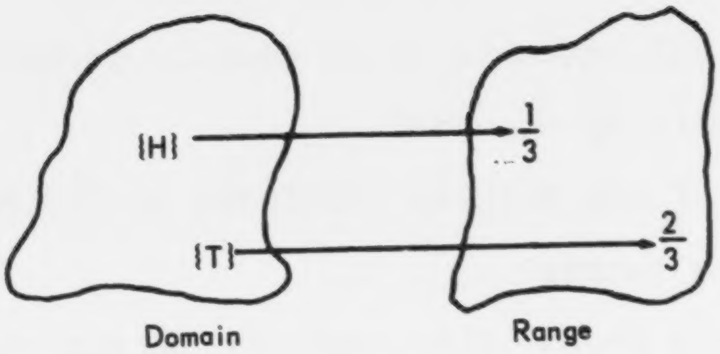


Figure 5.2

Notice that the outcomes seem to be those resulting from the toss of a coin. The images in the range could be the probabilities of the corresponding members of the domain. Is the coin a fair coin? Are the images between 0 and 1? If so, is the sum of the images equal to 1?

The mapping illustrated below shows the probabilities for certain events in a three-child family:

Domain	Range
{exactly three boys}	$\frac{1}{8}$
{exactly two boys}	$\frac{3}{8}$
{exactly one boy}	$\frac{3}{8}$
{no boys}	$\frac{1}{8}$
Sum	1

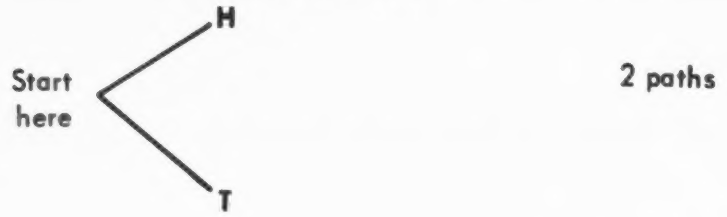
Question: Why is the probability of exactly two boys 3 times as great as the probability of exactly three boys?

Question: Make up an outcome set with 16 outcomes for the children in a four-child family. One of the outcomes, for example, will be BBGB. Illustrate the mapping of the outcomes, "exactly four boys," "exactly three boys," etc. onto their probabilities.

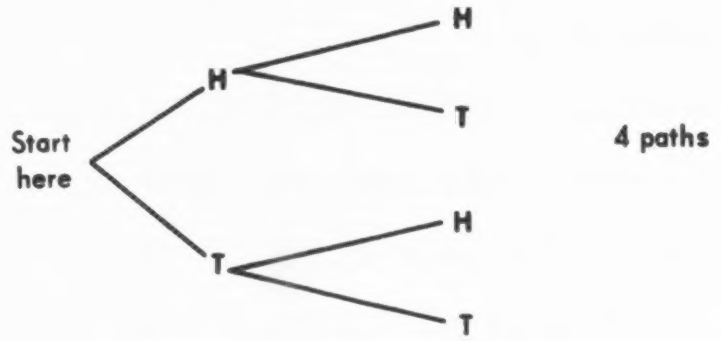
5.10 Counting with Trees

If an experiment involves several activities each having several alternatives, it is often a complicated task to count all of the possible outcomes and identify them. Below are some tree diagrams for coin tossing experiments. If you follow every path in the tree for an experiment you will discover all possible outcomes.

1. Tossing one coin



2. Tossing 2 coins



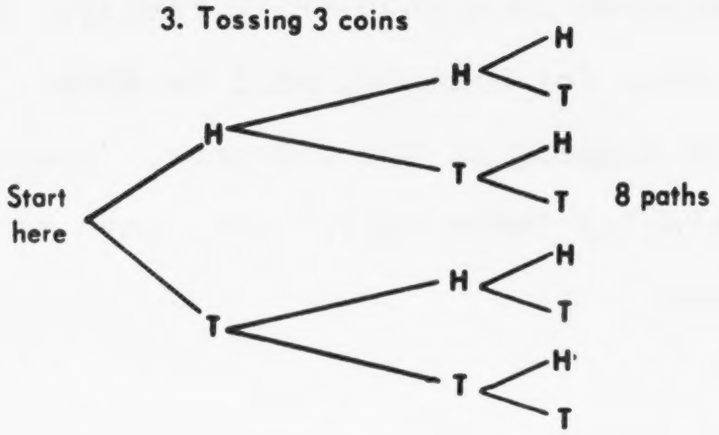


Figure 5.3

Exercise: Make a tree diagram for the possible outcomes of tossing first a die, then a coin, and then a thumbtack. You will have six branches to choose from at the starting point. Then each of these branches will have a certain number of branches, etc.

Question: How many paths are there?

5.11 Preview

The following ideas, which were illustrated in some of the preceding exercises, will be developed in more detail in your later study of mathematics:

1. If two events, A and B, have no outcomes in common then the probability that at least one of them occurs is the sum of the probabilities of the two events:

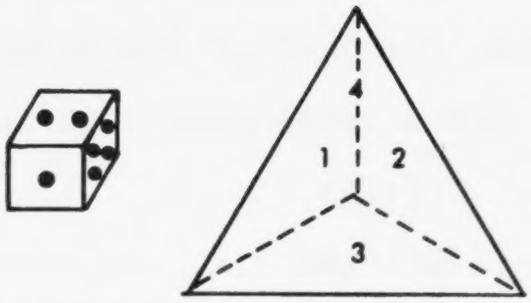
$$P(A \underline{\text{ or }} B) = P(A) + P(B)$$

2. If two events C and D, have outcomes in common, then the probability that at least one of them occurs is the sum of the probabilities of the two events minus the probability that both occur:

$$P(C \underline{\text{OR}} D) = P(C) + P(D) - P(C \underline{\text{and}} D)$$

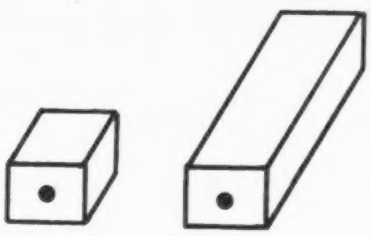
5.12 Exercises

1. (a)



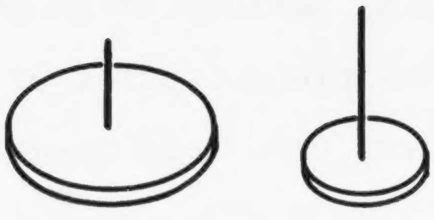
The tetrahedron has four faces. Imagine that on each face is a numeral from 1 to 4 respectively. Will the probability of the outcome 4 be greater for tossing the die or the tetrahedron? What are the probabilities in these two cases?

(b)



Which of the above dice would give the greater probability to the outcome 1?

(c)



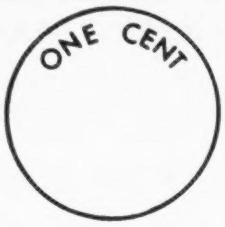
Which tack pictured above would be more likely to come to rest, pin-up? Why?

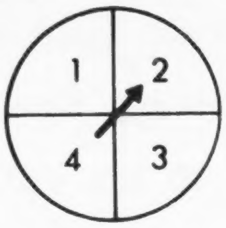
2. Use tree diagrams to make up an outcome set, using the simplest outcomes, for each of the following experiments:

(a) Toss a die and a tetrahedron, as in Exercise 1(a).

$\{(1,1), . . .\}$

(b) Toss a coin, spin the dial, select a vowel at random.

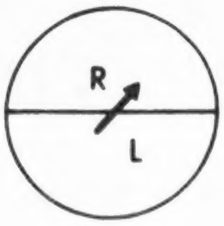




Think of each trial having an ordered triple as outcome, such as (H, 3,u).

3. -3 -2 -1 0 1 2 3

(a) Copy the above diagram and place a disk on the point labeled "0."



(b) Spin the dial. If the result is R, move the disk to the next point on the right. If the result is L, move the disk to the next integer on the left.

(c) Repeat moves until you reach 3 or -3. The outcome is whichever of 3 or -3 you reach first.

(d) Play the game several times and find the relative frequency of 3.

4. A bag contains 3 yellow marbles and 5 green marbles.

(a) If you select a marble without looking in the bag, what is the probability of selecting a yellow marble? a green marble?

(b) If you select a yellow marble on the first draw, and do not replace it, what is the probability of drawing a green marble on the second draw?

5. White rhinoceroses are very rare; the probability that one will be found among the rhinos of any African plain is $\frac{7}{5000}$. The Serengeti Plain in Africa has 10,000 rhinos. Estimate the number of Serengeti white rhinos.

6. If a letter is selected at random from the alphabet, what is the probability that the chosen letter is a vowel? a consonant?

7. On a page containing 2000 letters, about how many will be vowels?

8. An experiment is performed with outcome set $\{a,b,c\}$. If $P(a) = \frac{1}{6}$ and $P(b) = \frac{1}{4}$, than what is $P(c)$?

9. Try to explain the meaning of the probabilities in the following situations:

(a) An engineer says: The probability that the lamps we manufacture will burn more than 1000 hours is .05.

(b) According to Laplace (1749-1827), a famous French mathematician, the probability that a baby will be a girl is $\frac{22}{43}$.

(c) When you toss two dice the probability that you will get the sum 7 is equal to .17.

(d) A mathematician who has been consulted concerning inventory problems in a supermarket says: The probability that more than 1000 units of this kind will be sold during a day is .1.

(e) A meterologist says: When the weather conditions are what they are today, the probability that it will rain tomorrow is .15.

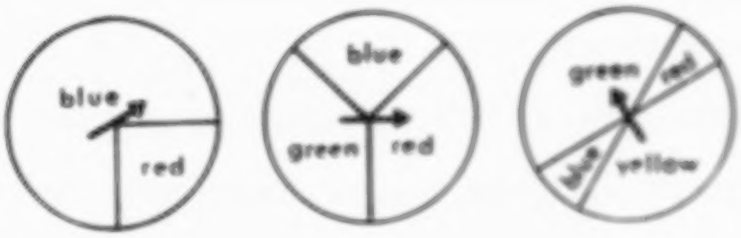
10. Use the probabilities given in Exercise 9 to answer the corresponding questions below.

(a) A city used 200 of the light bulbs described in 5 (a) to light one of its parking lots. If the lot opened on June 1 and the lights burned 24 hours a day, how many bulbs would probably burn out before July 12?

(b) If Babies Hospital registered 750 births in a two month period, how many would you expect to be girls?

(c) A pair of dice are tossed 50 times. On approximately what number of tosses will the sum be seven?

11.



(a) For which of the above dials is the probability of the spinner stopping on a red the greatest?

(b) Estimate the probability of the event "dial stops on red" for each of the above dials.

12. Make a tree diagram for Exercise 2.(b).

13. In a family with 6 children, what is the probability that all children are boys?

14. Toss a pair of dice of different colors (green and red).

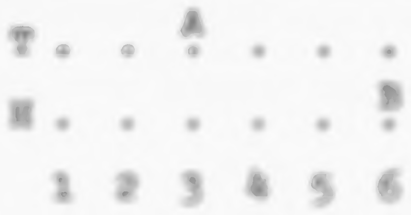
(a) What is the probability that at least one die will show 1 on the up-face?

(b) Draw a rectangular diagram of the 36 point outcome set and draw a closed curve enclosing the points for the event described in (a).

(c) What is the probability of the event, "green die 1 and red die 1?"

15. What is the probability that two people selected at random will both have birthday anniversaries on a Wednesday in 1968?

16. A coin and die are thrown, both fair. The outcome set for this experiment may be shown by the following:



(a) What outcome is represented by point A? point B?

(b) Place an oval around the points for the event:

"the die shows fewer than 3 dots."

(c) What is the probability for the event in (b)?

(d) What is the probability that the event in (b) does not occur?

(e) If in this experiment an event has 8 outcomes, what is its probability?

(f) If in an experiment there are exactly n possible outcomes, all equally likely, what is the probability of an event having r outcomes?

(g) What is the probability that a tail and an even number of dots show?

17. A die and 2 coins are thrown.

(a) Using a dot array represent all the outcomes of this experiment.

(b) What is the probability that 2 heads and an even number shows?

(c) What is the probability that 1 head, 1 tail, and an odd number shows?

5.13 Research Problems

In the diagram below the circles are called states and the routes for legally getting from one state to another are called paths. The numerals in circles A, B, C, D, and E indicate the number of paths from the start to the respective states.

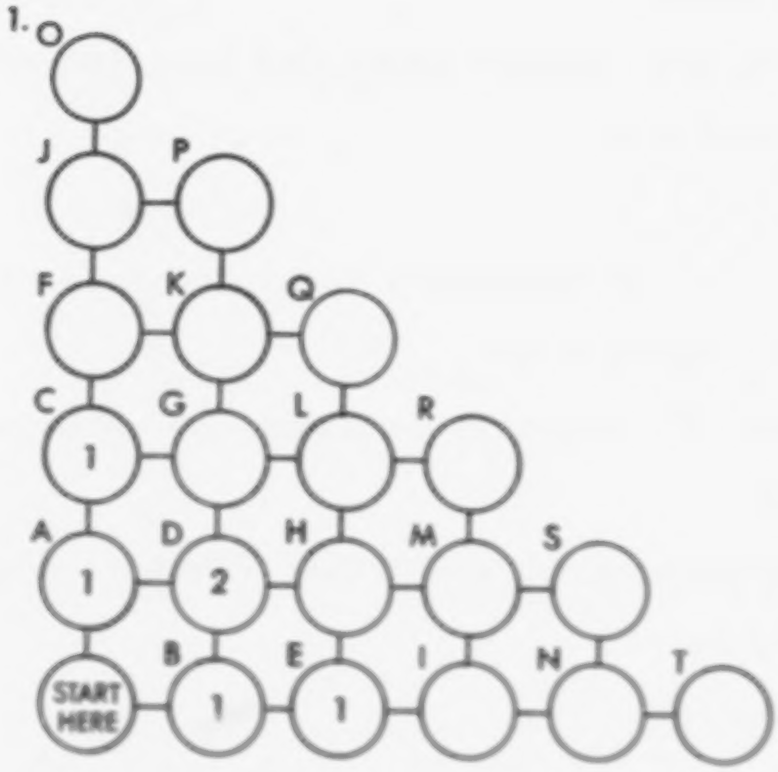


Figure 5.4

(1) Procedure

(a) Place a small disk on the lower left state labeled "start here."

(b) Toss a coin.

(c) If the coin lands heads-up, move to the next state on the right. If the coin lands tails-up, move to the next state above. (No moves to the left or down are allowed.)

(2) Experiment

(a) Toss a coin five times and make the proper moves on each toss. What state did you reach?

(b) Repeat the five-toss experiment 64 times and each time record your destination.

(c) What was the relative frequency for each destination?

(d) What do you notice about the location of your destinations?

(3) Experiment

(a) Record your destinations for a two-toss experiment with 32 repetitions.

(b) What was the relative frequency of each destination?

(c) What do you notice about the location of these destinations?

(4) Counting Paths

(a) Using the rules of our game, there is only one path to each of A, B, C and E but there are two paths to D. State G would have 3 paths, A-C-G, A-D-G, and B-D-G. Make a copy of the diagram of states and record the number of legal paths to each state inside the corresponding circles in the diagram.

(b) Except for the border states in the left column and the bottom row, each state has exactly two possible predecessors, the one below and the one to the left. Find a method of computing the number of paths to a state by using the number of paths to each predecessor.

(c) There are 2 one-toss paths, A and B. There are 4 two-toss paths, A-C, A-D, B-D and B-E.

How many three-toss paths are there? Four-toss?

(d) There are 32 five toss paths and 10 of these go to state Q. What is the probability of arriving at Q in five tosses, if we assume each path to be equally probable?

(e) Compute the probabilities for each state in the diagram.

2. (a) The Birthday Anniversary Problem

How large a group of people would you need so that the probability that at least two people in the group have the same birthday anniversary is $\frac{1}{2}$. (Any person born on February 29 will not be considered in this problem. And twins don't count.)

(b) A penny and a dime are tossed. You are informed that at least one turns up tails. What is the probability that both turn up tails? Plan and carry out an experiment of 100 tosses. How does the relative frequency of 2 tails compare with your theoretical answer?

(c) Consult Who's Who in America or a similar book and pick ten samples of 20 people in alphabetical order. Be sure to avoid overlap in your samples. This is then random enough for our purpose. How many of the ten samples contain a pair of people with the same birthday anniversary? Record the relative frequency of this occurrence.

5.14 Statistical Data

"Seventy-five per cent of the automobile accidents in this state happen within twenty miles of home."

Statements of this type, in which statistics are presumably used, are often made and often misinterpreted. A fragment of information, such as that mentioned above, leaves many important questions unanswered.

What is the source of this information? Were the accidents only those for which insurance claims were involved? Were they the accidents recorded in police records? Was the information acquired from some sample of accidents, or did it really include every actual accident? Over what period of time did these accidents occur? If the information was based on a sample, how was the sample chosen? What conclusions should we draw? Is it really more dangerous to drive near home, or is it possible that seventy five percent of all driving is done within twenty miles of home?

The above questions are related to the work done by statisticians. The statistician makes a science of gathering information, organizing it, analyzing it to see if there are any patterns, presenting it in the manner that will be most informative, making predictions on the basis of it, and verifying thes predictions.

In this section we will illustrate some ways of presenting information about events of various kinds, and ask you to gather certain data and present it in tables and graphs.

We will deal only with one aspect of statistics, namely descriptive statistics. And even in that area we will discuss only the presentation of data by graphical and tabular methods. How to analyze data by means such as averages, scattering, and probability will be discussed in future courses.

There are many ways of presenting and displaying numerical data. Tables, pictograms, and graphs of various kinds are the usual ways. Some of these are tables, line graphs, bar graphs, pictograms and circle graphs (pie-charts).

In this chapter we will study dot frequency diagrams, frequency histograms, and frequency polygons.

5.15 Presenting Data in Tables

During the summer playground programs, the children engaged in many activities, including basketball foul-shooting. Near the end of the program, the director organized a foul-shooting contest. A group of twenty boys and a group of twenty girls were selected as the first to participate. Each one had ten tries and the results were tabulated as below:

Table 4 Number of Baskets out of Ten Tries in a Foul-Shooting Contest

GIRLS

Contestant	Score	Contestant	Score
1	4	11	7
2	10	12	1
3	6	13	5
4	8	14	3
5	2	15	2
6	8	16	7
7	9	17	8
8	1	18	8
9	8	19	8
10	4	20	9

BOYS

Contestant	Score	Contestant	Score
1	1	11	7
2	3	12	8
3	3	13	1
4	5	14	3
5	9	15	1
6	7	16	7
7	9	17	10
8	7	18	6
9	6	19	6
10	9	20	10

The scores in Table 4 occur in the same order as that in which the players participated. As you look over the scores, try to answer the following questions.

(1) Did the girls do better than the boys?

(2) What is a good guess for the girls' average? boys' average?

(3) What would you estimate as the middle score for the girls? for the boys'?

(4) What score occured most frequently for the girls? the boys?

(5) How were the scores distributed? That is, were most of the scores either very high or very low; or did most of them cluster somewhere in between?

(6) In Table 5 below, the same scores are ranked by size. Now try to answer the same questions for Table 5.

Table 5 Number of Baskets in Ten Tries in a Foul-Shooting Contest

GIRLS	BOYS
1	1
1	1
2	1
2	3
3	3
4	3
4	5
5	6
6	6
7	6
7	7
8	7
8	7
8	7
8	8
8	9
8	9
9	9
9	10
10	10

Notice that Table 5 certainly gives more information about the middle score and the scores that would occur at about the $\frac{1}{4}$ mark and $\frac{3}{4}$ mark. You also get the feeling than neither group was unquestionably superior to the other.

The next table shows the frequency of scores grouped by intervals. This type of table is particularly effective when the data consists of large numbers of measures such as weights, lengths, or time intervals.

Table 6 Number of Baskets in Ten Tries in a Foul-Shooting Contest

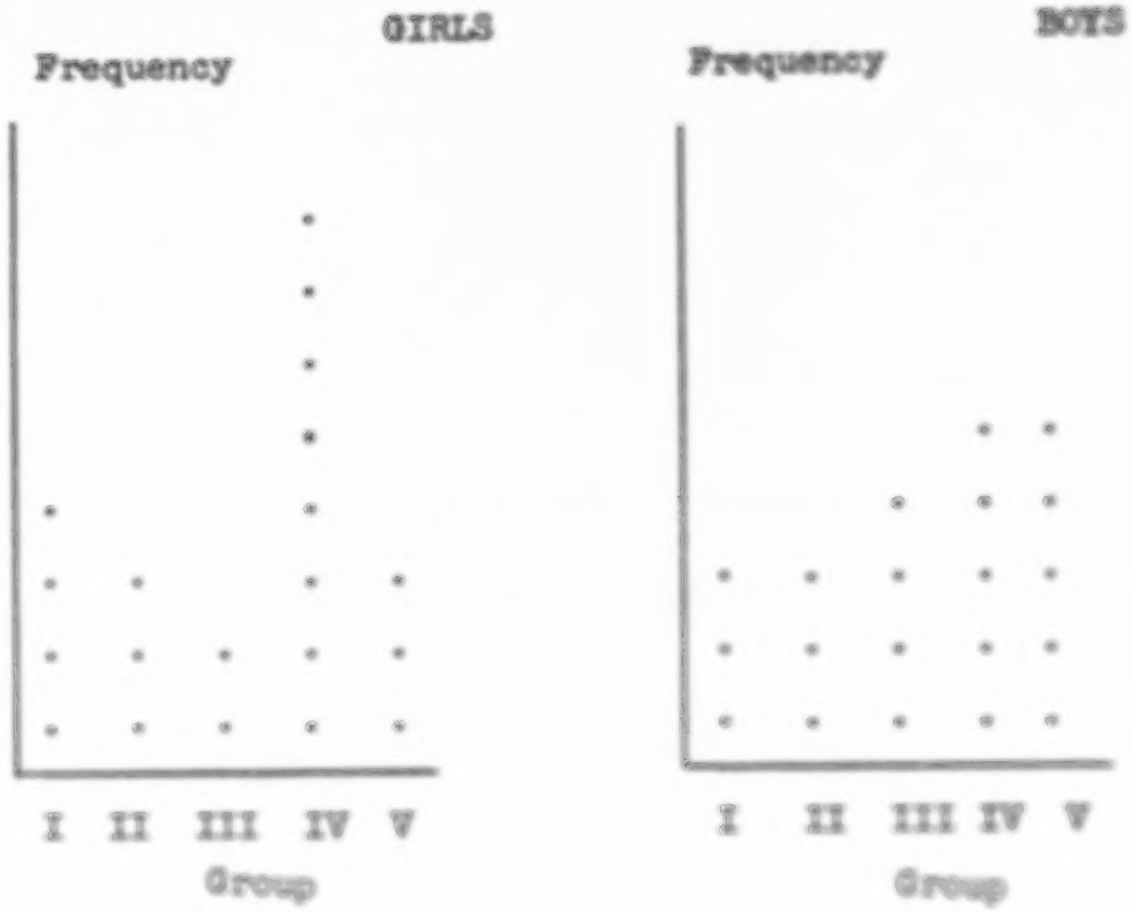
(Scores Grouped into Five Intervals)

		Frequency	
Group	Class Interval	Girls	Boys
I	1 - 2	4	3
II	3 - 4	3	3
III	5 - 6	2	4
IV	7 - 8	8	5
V	9 - 10	3	5

Below are two dot frequency diagrams for the same information represented in the Tables 4, 5, and 6.

Graph 3

DOT FREQUENCY DIAGRAM



5.16 Exercises

1. Discuss the following:
   (a) The advantage of ranking data as in Table 5 or 6, or Graph 3.
   (b) The advantages of grouping data into class intervals.
2. Find the number such that (use information in Table 5)
   (a) 25% of the scores are less than or equal to the number;
   (b) half of the scores are less than or equal to the number;
   (c) 75% of the scores are less than or equal to the number:

3. Do the following for the set of test marks below:
   (a) Rank the marks according to numerical order.
   (b) Group the marks into intervals from 61 to 65; 66 to 70; etc.
   (c) Make a frequency table showing the frequency for each interval.
   (d) Make a dot frequency diagram showing the frequency for each interval.

   Test Marks: 73, 67, 72, 88, 75, 89, 79, 81, 70, 93, 76, 79, 82, 98, 90, 72, 70, 83, 78, 85, 73, 84, 92, 80, 69, 81, 78, 90, 93, 76, 78, 62, 83, 78 and 88.

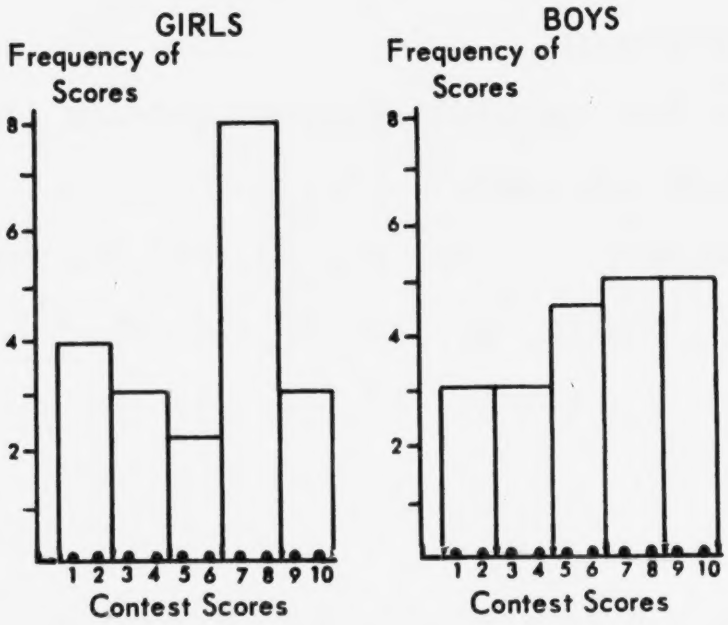
5.17 The Frequency Histogram and the Cumulative Frequency Histogram

The frequency histogram is very similar to the dot frequency diagram. In place of the vertical columns of dots there are rectangles with width equal to the length of the group interval. The height of the rectangle is the frequency in the interval. Study the histograms below and compare them with the dot frequency diagrams of Graph 3 which present the same data.

Graph 4

FREQUENCY HISTOGRAMS

FREQUENCY OF SCORES IN FOUL-SHOOTING CONTEST

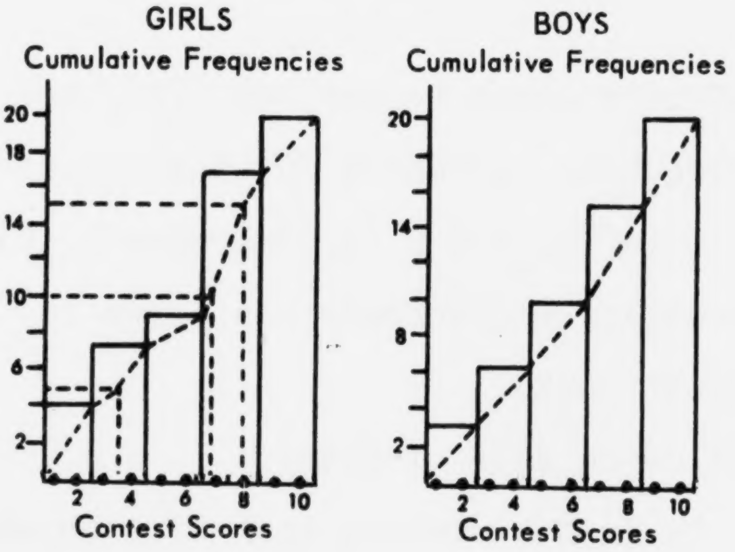


The cumulative frequency histogram is similar to the frequency histogram except that the second rectangle has height equal to the sum of the heights of the first two in the frequency histogram, the third is the sum of the first three, etc. A table is included below which tabulates the cumulative frequencies to help interpret the cumulative frequency histograms.

TABLE 7 CUMULATIVE FREQUENCY TABLE FOR FOUL-SHOOTING CONTEST SCORES

Class Interval	Frequency		Cumulative Frequency	
	Girls	Boys	Girls	Boys
1 - 2	4	3	4	3
3 - 4	3	3	7	6
5 - 6	2	4	9	10
7 - 8	8	5	17	15
9 - 10	3	5	20	20

GRAPH 5 CUMULATIVE FREQUENCY HISTOGRAMS FOR FOUL-SHOOTING CONTEST SCORES



In both of the cumulative frequency histograms there is a dotted segment connecting the upper right corners of the rectangles. This set of segments is called a cumulative frequency polygon. It is helpful in determining the number below which 25% of the scores fall. This number is called the first quartile. It is likewise helpful in finding the comparable number for 50% of the scores or in fact any particular percent of the scores.

Notice the horizontal dotted segments, going from 5, 10, and 15 on the vertical scale over to the polygon and then down to the horizontal scale. These determine the numbers which 25%, 50%, and 75% of the scores are less than or equal to. Other names for these numbers are first quartile, median, and third quartile respectively. They are very useful in classifying scores for comparison purposes.

5.18 Exercises

1. Use the set of test marks in Section 5.16, Exercise 3 (d).
    (a) Make a frequency histogram for the set of test marks, grouped as in Section 5.16, Exercise 3 (b).
    (b) Make a cumulative frequency histogram for the set of grouped test marks.
2. Gather the following sets of data:
    (a) The heights to the nearest inch of each member of your class.
    (b) The ages to the nearest month of the members of your class.

(c) The number of cars passing a certain point in some street during 20 five-minute intervals.

3. Present the data of Exercise 1 in the following ways:
   (a) a cumulative frequency table with a separate entry for each measure;
   (b) a cumulative frequency table with the data grouped into intervals;
   (c) a cumulative frequency polygon based on the table of part a.

5.19 Summary

1. This chapter introduced several mathematical methods of predicting the outcome of activities in situations involving uncertainty. In the fish count problem it is impractical to do more than estimate on the basis of incomplete knowledge. In die tossing, the outcome of a given trial can never be known in advance.

2. To assist in making good estimates or predictions, we performed a limited number of trials and observed the relative frequency of the various possible outcomes. We found that for a given experiment, the relative frequencies tended to stabilize as the number of trials increased.

3. On the basis of this stability of relative frequency, we made predictions of the likelihood or probability of events. The probability of an event, like the relative frequency, is a number assigned to the event. The number is

(A) 0, 1 or a number between 0 and 1,

(B) 0 for an impossible event;

(C) 1 for a certain event.

4. Furthermore, we found that probabilities and relative frequencies had the following properties:

(A) The sum of the probabilities (relative frequencies) of the outcomes in an outcome set is 1.

(B) If two events have no outcomes in common, the probability that one of the two will occur on a given trial is equal to the sum of the probabilities of the individual events.

(C) If an experiment has $\underline{n}$ equally probable outcomes and an event has $\underline{s}$ outcomes, the probability of the event is s/n.

5. The presentation of results is an important part in analysis of data collected from experiments. We saw how to graph the results of experiments by

(A) dot frequency diagrams,

(B) frequency histograms,

(C) cumulative frequency histograms,

(D) frequency polygons,

(E) cumulative frequency polygons.

5.20 Review Exercises

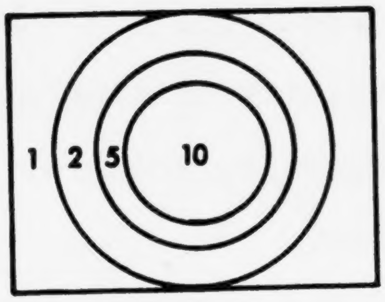
1. List the members of an outcome set for each of the following experiments:

(a) Select two means of transportation from {bus, train, plane}.

(b) A dodecahedron (twelve-faced polyhedron) with faces numbered from 1 to 12 is rolled and the numeral on the up-face is observed.

(c) A pair of vowels is selected from the alphabet.

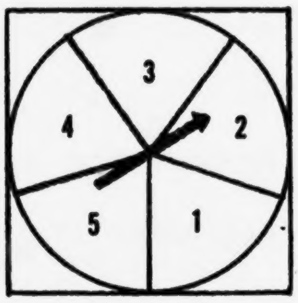
(d) Two darts are thrown at a target with four scoring possibilities:

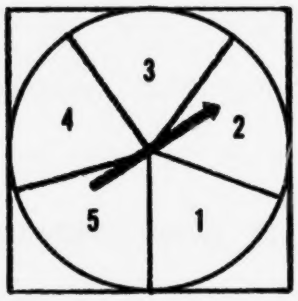


(e) Three tags are selected from a box containing five blue tags and two red tags.

(f) Each of three people vote for Jones or Smith (but not both).

2. Two dials with sectors numbered from 1 to 5 are spun:





(a) Tabulate the outcome set.

(b) How many ordered pairs are in the outcome set?

(c) Assuming that each ordered pair in the outcome set is equally likely, what is $P(\{(2,5)\})$?

(d) What is the probability that both dials will yield an even number?

(e) What is the probability that at least one of the dials will yield an even number?

(f) Make a rectangular arrangement of dots to represent the outcome set, as in Section 5.8 Exercise 3.

(g) Draw a line through the dots of the rectangular arrangement for which the sums for the outcome are each six. Repeat for the sums five and seven. What is the probability of each of the above sums?

(h) Circle the dots for which at least one dial yields an even number.

3. Select two pages of a magazine article and separate the text into sets of ten lines.

(a) Find the relative frequency of the letter e, for each set of ten lines.

(b) Find the relative frequency of the letter x, for each of ten lines.

(c) Compare the relative frequencies of e and x.

(d) Among the samples tested, were the relative frequencies for e fairly uniform? Answer the same question for x.

(e) What predictions could you make on the basis of the above investigation?

4. A coin and die are tossed simultaneously.

(a) Tabulate an outcome set which pairs each of the outcomes for the die with each for the coin.

(b) Assume that each simple outcome is equally likely.

(c) What is the probability that the die will show six?

(d) What is the probability that the die will not show six?

(e) What is the sum of the probabilities in (c) and (d)?

(f) For any event, E, what is the following sum,

$$P(E) + P(\text{not } E)?$$

(g) What is the probability that the die will show six, given that the coin lands heads?

(h) Does the probability of the outcome six for the die depend on the event that the coin landed heads?

(i) What is the probability that the coin lands heads and the die shows six?

(j) Is the probability of the event described in (i) equal to the product of the probabilities for the coin landing heads and the die showing 6?

5. Describe two events, A and B, from the experiment in Exercise 4 that have no outcomes in common.

(a) What is $P(A)$? $P(B)$?

(b) What is $P(A) + P(B)$?

(c) What is the probability of the event, A occurs or B occurs?

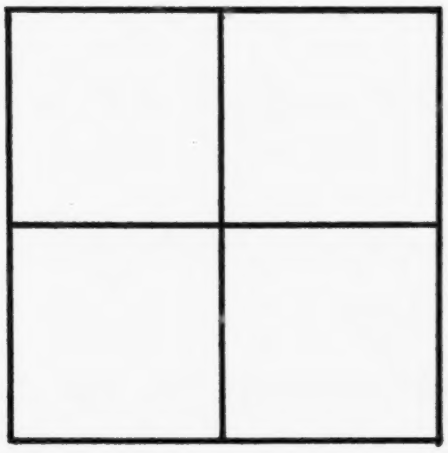
(d) What generalization is suggested by the answers to (a), (b) and (c)?

6. Make a table showing the number of children in the family of each student in your class. Then make a table showing the relative frequency of one-child families, two-child families, etc. The illustrative table below shows that for a class of twenty students there were 5 one-child families so that the relative frequency for one-child families (in this sample) was $\frac{5}{20}$ or $\frac{1}{4}$.

NUMBER OF CHILDREN per FAMILY IN
A SAMPLE OF TWENTY FAMILIES

Number of Children	Frequency	Relative Frequency
1	5	$\frac{1}{4}$
2	4	$\frac{1}{5}$
etc.	etc.	etc.

7. Make a similar chart for the distances from home to school for each student in your class, using the number of blocks as a measure. Group the data into class intervals. For example, all students living from 0 to 4 blocks from school might be grouped together, then 5 to 9, etc. Make a histogram and frequency polygon for this data.

8. 

In an agricultural experiment a field is divided into four square regions as pictured above. Two of these are selected at random and given a special treatment. What is the probability that the selected squares are

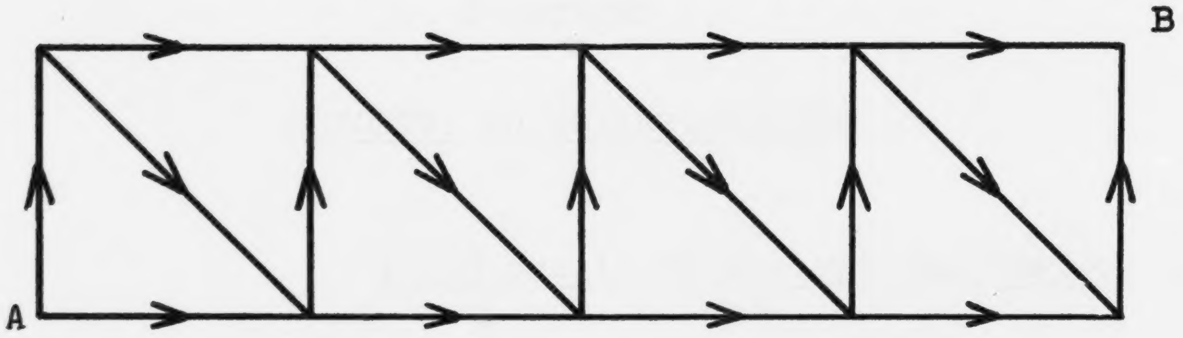
(a) in the same row?

(b) in the same column?

(c) in the same row or the same column?

(d) one in each row and one in each column?

9.



In tracing a path in the above network from A to B, a selection of direction is made at each corner by tossing a coin. In how many ways can you go from A to each of the different corners?

10. In how many ways can three cars, A, B and C be parked in a row? If the cars are parked at random, what is the probability that A and B are next to each other?

CHAPTER 6

MULTIPLICATION OF INTEGERS

6.1 Operational Systems (W,·) and (Z,·)

In Chapter 4 we learned how to add and subtract integers. It is natural to ask how integers should be multiplied.

The operational system (Z,+) is, for two reasons, the natural extension of (W,+). First, since the set of integers includes the set of whole numbers as a subset, addition of non-negative integers is the same as addition of whole numbers. Second, many of the important properties of (W,+) are also properties of (Z,+). It seems reasonable to expect multiplication in Z to be a similar extension of multiplication in W -- multiplication of non-negative integers will be the same as multiplication of whole numbers and (Z,·) should have properties similar to the following properties of (W,·).

1. For all whole numbers a and b, $a \cdot b = b \cdot a$.
   (Commutative Property of Multiplication)
   For example: $3 \cdot 7 = 7 \cdot 3$.
2. For all whole numbers a, b, and c,
   $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
   (Associative Property of Multiplication)
3. For every whole number a, $1 \cdot a = a \cdot 1 = a$.
   (1 is a Multiplicative Identity in W)
4. For every whole number a, $a \cdot 0 = 0 \cdot a = 0$.
   (Multiplication Property of Zero)

For example: $3 \cdot 0 = 0 \cdot 3 = 0$.

5. If a, b, and c are whole numbers with $c \neq 0$, and $c \cdot a = c \cdot b$, then $a = b$.

   (Cancellation Property of Multiplication)

   For example: If $7 \cdot a = 7 \cdot 13$, then $a = 13$.

There is one property of the operational system $(W,+,\cdot)$ which relates the operations of addition and multiplication. This property is illustrated in the following example.

Suppose we compute the product 7 x 13 in the usual way:

$$\begin{array}{r} 13 \\ \underline{x7} \\ 91 \end{array}$$

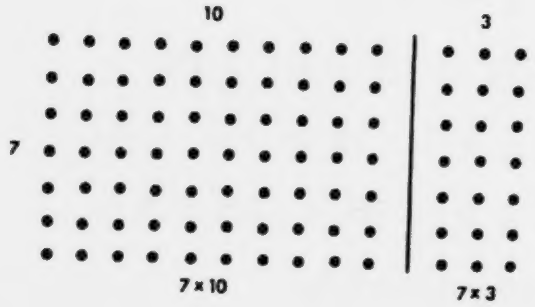
In this computation, we have actually computed two simpler products:

$$7 \times 3 = 21 \quad \text{and} \quad 7 \times 10 = 70$$

and then computed their sum

$$21 + 70 = 91$$

The reason this works is easy to understand if we picture the product 7 x 13 as a rectangular array that has been split into two arrays:



On the left we have a 7 x 10 array and on the right a 7 x 3 array. The number of elements in the array does not change by the splitting, so we have

$$7 \cdot (10 + 3) = (7 \cdot 10) + (7 \cdot 3).$$

Similarly, we know that

$$7 \cdot (4 + 6) = (7 \cdot 4) + (7 \cdot 6),$$

$$13 \cdot (98 + 2) = (13 \cdot 98) + (13 \cdot 2).$$

These examples are instances of the sixth important property of $(W,+,\cdot)$.

6. For any whole numbers a,b, and c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

(Distributive Property of Multiplication over Addition)

We should also like the distributive property to apply in $(Z,+,\cdot)$.

6.2 Exercises

1. For each of the following state the property for multiplication of whole numbers that justifies the equality.
   (a) $87 \times 1 = 1 \times 87$
   (b) $87 \times 1 = 87$
   (c) $(98 - 97) \times 46 = 46$
   (d) $5 \times (2 \times 83) = (5 \times 2) \times 83$
   (e) $(25 \times 38) \times 4 = (38 \times 25) \times 4$
   (f) $(38 \times 25) \times 4 = 38 \times (25 \times 4)$
2. Without computing justify:
   (a) $(43 \times 28) \times 76 = (76 \times 43) \times 28$

(b) $87 \times (43 \times 76) = (87 \times 76) \times 43$

(c) $8 \times (69 \times 25) = 69 \times (25 \times 8)$

3. State the commutative property for addition of whole numbers.

4. State the associative property for addition of whole numbers.

5. What is the identity element for addition of whole numbers?

6. What is the identity element (if there is one) for each of the following systems?

(a) $(Z_5,+)$ (e) $(Z,+)$

(b) $(Z_5,\cdot)$ (f) $(W,+)$

(c) $(Z_6,+)$ (g) $(W,\cdot)$

(d) $(Z_6,\cdot)$ (h) $(Z,-)$

7. Compute each of the following:

(a) $8 \times (9 \times 7)$ (d) $(8 \times 7) \times 9$

(b) $9 \times (8 \times 7)$ (e) $(47 \times 73) + (47 \times 27)$

(c) $7 \times (9 \times 8)$ (f) $(47 \times 73) - (47 \times 27)$

*8. Using the properties of this section, prove each of the following, given that r, s, t are whole numbers.

(a) $(r \cdot s) \cdot t = (r \cdot t) \cdot s$

(b) $(r \cdot s) \cdot t = (t \cdot s) \cdot r$

(c) $r \cdot (s \cdot t) = (r \cdot t) \cdot s$

(d) $r \cdot (s \cdot t) = s \cdot (t \cdot r)$

For example, exercise (a) may be done as follows:

$(r \cdot s) \cdot t = r \cdot (s \cdot t)$ Multiplication of whole numbers is associative.

$= r \cdot (t \cdot s)$ Multiplication of whole numbers is commutative.

$= (r \cdot t) \cdot s$ Multiplication of whole numbers is associative.

9. From your experience with multiplication of whole numbers what seems to be true if the factors are ordered and grouped differently? (The generalization referred to here is sometimes called "the rearrangement property for multiplication of whole numbers.")

10. Compute:

    (a) $7 \times (20 + 7)$

    (b) $(7 \times 20) + (7 \times 7)$

    (c) $(23 \times 87) + (23 \times 13)$

    (d) $(76 \times 38) + (24 \times 38)$

    (e) $(47 \times 39) - (47 \times 29)$

    (f) $(37 \times 43) - (27 \times 43)$

    (g) $(6\frac{1}{2} \times 8) + (6\frac{1}{2} \times 12)$

    (h) $(6\frac{1}{2} \times 3\frac{1}{2}) + (6\frac{1}{2} \times 6\frac{1}{2})$

11. Is it true that $5 + (2 \times 4) = (5 + 2) \times (5 + 4)$?

12. Is addition distributive over multiplication in $(W,+,\cdot)$?

6.3 Multiplication for Z

In order to define multiplication as an operation in Z, we must show how to assign to each ordered pair (a,b) of integers a third integer c called "the product of a and b." We will use the definition of multiplication for whole numbers and the six properties we want preserved as guides to the rule of assignment for "·" in Z. We also want our definition to make sense in situations where the integers have applications to real life problems. Under these circumstances, there are four cases which must be considered in making our definition:

1. Both a and b are positive.
2. Both a and b are negative.
3. a is positive and b is negative.
4. a or b (or both) is zero.

Question: Why is it unnecessary to consider the case "a negative, b positive"?

We already know how to multiply positive integers and how to multiply by zero. For example,

$$3 \times 4 = 12$$

$$11 \times 14 = 154$$

$$8 \times 0 = 0$$

$$0 \times 3 = 0$$

These examples suggest that we should make the following definitions: The product of two positive integers is the unique positive integer whose absolute value is the product of the absolute values of the factors. If a is either a positive integer or zero, $a \cdot 0 = 0 \cdot a = 0$.

6.4 Multiplication of a Positive Integer and a Negative Integer

Every integer is either positive, zero, or negative. In other words, for every integer n, exactly one of these conditions must hold:

$$0 > n, \quad 0 = n, \quad \text{or } n < 0.$$

Let us now write a few computations that may suggest what the product of a positive integer and a negative integer should be.

$$
\begin{aligned}
3 \times 3 &= 9 \\
3 \times 2 &= 6 \\
3 \times 1 &= 3 \\
3 \times 0 &= 0 \\
3 \times (-1) &= \square \\
3 \times (-2) &= \square \\
3 \times (-3) &= \square
\end{aligned}
$$

In this column of equalities, the second factor decreases by 1 as we move down. The corresponding products decrease by 3.

This list suggests that the products for the last three lines should be -3, -6 and -9, if the products are to continue to decrease by 3. It appears that the product of a positive integer and a negative integer should be negative. Furthermore, the absolute value of the product should again be the same as the product of the absolute value of the factors. Since we want multiplication in Z to be commutative, the product of a negative integer and a positive integer should be computed in the same way.

Later we shall give other reasons for adopting this definition. Let us now see some illustrative examples.

Example 1. Compute $(-8) \times 7$.

$$
\begin{aligned}
|(-8) \times 7| &= |-8| \times |7| \\
&= 8 \times 7 \\
&= 56
\end{aligned}
$$

Since -8 is negative and 7 is positive, $(-8) \times 7$ is a negative integer. Hence $(-8) \times 7 = -56$.

Example 2. Compute $9 \times (-6)$.

$$\begin{aligned} |9 \times (-6)| &= |9| \times |-6| \\ &= 9 \times 6 \\ &= 54 \end{aligned}$$

Therefore, $9 \times (-6) = -54$.

Example 3. Compute $(4 \times (-3)) \times 2$.

$$\begin{aligned} (4 \times (-3) \times 2 &= (-12) \times 2 \\ &= -24 \end{aligned}$$

6.5 The Product of Two Negative Integers

The only products remaining to be considered are those involving two negative integers or a negative integer and zero. Once again, let us try to obtain a clue by recognizing a pattern.

$$\begin{aligned} (-3) \times 3 &= -9 \\ (-3) \times 2 &= -6 \\ (-3) \times 1 &= -3 \\ (-3) \times 0 &= \square \\ (-3) \times (-1) &= \square \\ (-3) \times (-2) &= \square \\ (-3) \times (-3) &= \square \end{aligned}$$

In this column of equalities, the second factor is again being reduced by 1 in moving down. The corresponding products are increasing by 3. This list then suggests that the last four products should be 0, 3, 6 and 9, if the products are to continue to increase by 3.

These examples suggest the following definitions: The product of a pair of negative integers is the unique positive integer which has absolute value equal to the product of the absolute values of the factors. For every negative integer a, $a \cdot 0 = 0 \cdot a = 0$. Later we shall give other reasons for adopting these definitions.

The definition of multiplication for Z which has been suggested by the patterns in this and the preceding sections can be summarized as follows:

For any integers r and s,

1. $|r \cdot s| = |r| \cdot |s|$.
2. If r and s are both negative or both positive, $r \cdot s$ is positive.
3. If one of r, s is positive and the other negative, $r \cdot s$ is negative.
4. $r \cdot 0 = 0 \cdot r = 0$.

With the above definition as rules for the assignment, multiplication is an operation on Z. That is, for each ordered pair (a,b) of integers there is a unique integer $c = a \cdot b$ called "the product of a and b." Furthermore, it can be shown that the six properties of $(W,+,\cdot)$ stated in Section 6.1 are also properties of $(Z,+,\cdot)$, when "$\cdot$" is defined in this way.

The general rules for multiplication of integers may be clarified by the following illustrative examples.

Example 1. Compute $(-3) \times (-4)$.

$$\begin{aligned} |(3) \times (-4)| &= |-3| \times |-4| \\ &= 3 \times 4 \\ &= 12. \end{aligned}$$

Since -3 and -4 are both negative, the product is positive. Hence, $(-3) \times (-4) = 12$.

Example 2. Compute $((-7) \times (-2)) \times (-3)$.

$$\begin{aligned} ((-7) \times (-2)) \times (-3) &= 14 \times (-3) \\ &= -42. \end{aligned}$$

Example 3. Compute $(-9) \times (6 \times (-4))$.

$$\begin{aligned} (-9) \times (6 \times (-4)) &= (-9) \times (-24) \\ &= 216. \end{aligned}$$

6.6 Exercises

1. Compute:
   - (a) $(-20) \times 27$
   - (b) $33 \times (-37)$
   - (c) $27 \times (-20)$
   - (d) $(-8) \times (-14)$
   - (e) $(-14) \cdot (-8)$
   - (f) $(-37) \times 33$
2. Compute:
   - (a) $-5 \times (2 \times (-47))$
   - (b) $((-43) \times (-4)) \times (-25)$
   - (c) $(10 \times (-6)) \times 5$
   - (d) $10 \times ((-6) \times 5)$
   - (e) $((-5) \times 2) \times (-47)$
   - (f) $(-43) \times ((-4) \times (-25))$
3. Compute:
   - (a) $((-17) \times (-7)) + ((-17) \times (-3))$

(b) (-17) x (-7 + (-3))

(c) ((-38) + 28) x (-37)

(d) ((-83) x 67) + ((-17) x 67)

(e) (-100) x 67

(f) ((-38) x (-37)) + (28 x (-37))

(g) ((-27) + 73) x (27 + 73)

4. Without computing, determine whether the following products are positive, negative, or zero.

(a) (-6)(-3)(10)(-8)

(b) (5)(11)(7561)(-2)(-15)

(c) (-7)(-7)(-7)(-3)(-4)(-5)

(d) $(-2)^{10}$

(e) $(-2)^{17}$

6.7 Multiplication of Integers through Distributivity

In Section 6.1 we thought it reasonable to require that (Z,+,·) retain the distinctive properties of (W,+,·). In order to extend the close relationship between (W,+) and the non-negative integers, we assumed that the product of two positive integers is a positive integer. Then by observing patterns of multiplication, we were led to definitions of a · b in the cases where one or both factors are negative or zero. We found that these definitions did preserve the desired properties.

Are there other possible ways to define multiplication in **Z** and still retain those properties? Could such alternative definitions lead to results differing from those we have already obtained? For example, could r · 0 = r for every integer r?

Could the product of two negative integers turn out to be a negative integer? (For instance, could $(-7) \cdot (-13) = -91$?)

In this section we shall show that if "$\cdot$" is assumed to be a commutative, associative, and distributive (over +) operation, the customary rules for computing products are actually forced on us.

Let us begin by stating a basic assumption which we have been using over and over. To illustrate this assumption, which we shall soon name, consider the easy computation

$$\begin{aligned}(2 + 3) + 4 &= 5 + 4\\ &= 9.\end{aligned}$$

The symbols "2 + 3" and "5" both name the same number so we feel free to replace "2 + 3" by "5." In the last step we replaced "5 + 4" by "9" because they both name the same number.

In mathematics we frequently replace one name for an object by another name for the same object, assuming that this kind of replacement is permitted. This assumption can be stated as follows: The mathematical meaning of an expression is not changed if in this expression one name of an object is replaced by another name for the same object. This assumption will be called the Replacement Assumption or simply Replacement. We shall be making frequent use of this assumption without mentioning it.

Why must $r \cdot 0 = 0$ for every integer? If r is positive or zero, we define $r \cdot 0 = 0 \cdot r = 0$ so that multiplication of non-negative integers is the same as multiplication of whole numbers. But what about $(-5) \cdot 0$, $(-32) \cdot 0$, $(-2162) \cdot 0$, etc.?

Certainly

$$(-5) \cdot 0 = (-5) \cdot 0. \tag{1}$$

Since $0 + 0 = 0$, statement (1) implies that

$$-5(0 + 0) = (-5) \cdot 0, \tag{2}$$

replacing "0" by "0 + 0." Because of the distributive property of multiplication over addition, $-5(0 + 0) = (-5 \cdot 0) + (-5 \cdot 0)$. Thus we can replace the left side of (2) to get

$$(-5 \cdot 0) + (-5 \cdot 0) = (-5) \cdot 0. \tag{3}$$

By the additive property of 0, $(-5) \cdot 0 = (-5) \cdot 0 + 0$ and (3) is equivalent to

$$(-5 \cdot 0) + (-5 \cdot 0) = (-5 \cdot 0) + 0. \tag{4}$$

Since addition of integers has the cancellation property, (4) implies that

$$(-5) \cdot 0 = 0.$$

The above argument shows that if $(Z,+,\cdot)$ is to satisfy the properties of $(W,+,\cdot)$, in particular the distributive and cancellation properties, then $(-5) \cdot 0$ must be defined to be 0. An obviously true statement, (1), leads to a chain of true statements, each of which follows from its predecessor because of a property we demand of $(Z,+,\cdot)$.

The argument that is given seems to apply only to the product $(-5) \cdot 0$. Since we require multiplication of integers to be commutative, $0 \cdot (-5)$ must also be 0. But what about $(-32) \cdot 0$, $(-2162) \cdot 0$, and, in general, $r \cdot 0$? If you study the argument given for $(-5) \cdot 0$, you will see that the argument can be repeated in the same form for the other products.

T1. For every integer *r*, $r \cdot 0 = 0$.

$r \cdot 0 = r \cdot 0$	Multiplication is an operation in Z.
$r \cdot (0 + 0) = r \cdot 0$	$0 + 0 = 0$, since 0 is additive identity.
$r \cdot (0 + 0) = (r \cdot 0) + 0$	$(r \cdot 0) + 0 = r \cdot 0$, since 0 is additive identity.
$(r \cdot 0) + (r \cdot 0) = (r \cdot 0) + 0$	Distributivity.
$r \cdot 0 = 0$	Cancellation for Addition.

This generalization or theorem says that the product of two integers is zero whenever one of the integers (or both) is zero. You recall that in Section 6.5 we defined $r \cdot 0 = 0 \cdot r = 0$. T1 shows that if we make the desired assumptions about "$\cdot$" there is really no choice in the definition of $r \cdot 0$. It must be zero! These desired assumptions place further restrictions on the rules for computing products. The product of a positive integer and a negative integer must be a negative integer. For example, $5 \cdot (-10) = -50$, $32 \cdot (-15) = -480$, and $2162 \cdot (-4) = -8648$.

Because of T1,

$$5 \cdot 0 = 0 \tag{1}$$

Since $10 + (-10) = 0$, statement (1) implies that

$$5 \cdot (10 + (-10)) = 0, \tag{2}$$

replacing "0" by "$(10 + (-10))$." Because of the distributive property of multiplication over addition, $5 \cdot (10 + (-10)) = (5 \cdot 10) + (5 \cdot (-10))$. Thus we can replace the left side of (2) to get

$$(5 \cdot 10) + (5 \cdot (-10)) = 0. \tag{3}$$

We know that $5 \cdot 10 = 50$. Substituting in (3) yields

$$50 + (5 \cdot (-10)) = 0. \tag{4}$$

Statement (4) implies that $5 \cdot (-10)$ must be the additive inverse of 50, or

$$5 \cdot (-10) = -50.$$

Again you can see that this argument -- though valid only for $5 \cdot (-10)$ -- can be mimicked for any other product of a positive and negative integer. In fact, a variation of this argument can be used to show that the product of two negative integers must be a positive integer. Consider, for example, the product $(-5) \cdot (-10)$.

$$\begin{aligned}
(-5) \cdot 0 &= 0 && \text{T1} \\
-5 \cdot (10 + (-10)) &= 0 && 0 = 10 + (-10) \\
((-5) \cdot (10)) + ((-5) \cdot (-10)) &= 0 && \text{Distributivity} \\
-50 + ((-5) \cdot (-10)) &= 0 && \text{Proven above.}
\end{aligned}$$

Therefore,

$$(-5) \cdot (-10) = 50 \qquad \text{50 is the additive inverse of } -50.$$

T1 and the other arguments of this section show that if $(Z,+,\cdot)$ is to retain the desirable properties of $(W,+,\cdot)$, the definition of multiplication stated in Section 6.5 is the only definition possible.

6.8 Exercises

1. Compute:

(a) $(-10) \cdot 7$

(b) $10 \cdot (-7)$

(c) $(-8) \cdot (-6)$

(d) $8 \cdot 6$

(e) $(-13) \cdot 12$

(f) $13 \cdot 12$

(g) $19 \cdot (-22)$

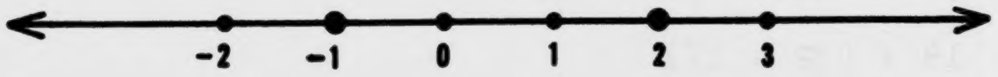
(h) $(-19) \cdot 22$

2. Compute:
   - (a) $10(18 - 27)$
   - (b) $(10 \cdot 18) - (10 \cdot 27)$
   - (c) $(14 \cdot 10) + (14 \cdot 5)$
   - (d) $14 \cdot (32 - 17)$
   - (e) $(14 \cdot 32) - (14 \cdot 7)$
   - (f) $10 \cdot (18 + (-27))$
   - (g) $(10 \cdot 18) + (10 \cdot (-27))$
   - (h) $14 \cdot (32 + (-17))$
   - (i) $(14 \cdot 32) + (14 \cdot (-17))$

3. Compute each of the following if $r = -4$, $s = -7$ and $t = 9$.
   - (a) $r + s$
   - (b) $r + (-s)$
   - (c) $r - s$
   - (d) $r(s + t)$
   - (e) $r(s - t)$
   - (f) $(rs) - (rt)$
   - (g) $(rs)t$
   - (h) $(rt)s$
   - (i) $(st)r$
   - (j) r^2
   - (k) r^3
   - (l) $(r^2)s$
   - (m) $r - t$
   - (n) $-(r - t)$
   - (o) $-r + t$
   - (p) $-2r + (-3t)$
   - (q) $r^2 + s^2$
   - (r) $r^2 - s^2$

4. Find the solution set from the set of integers for each of the following conditions.
   - (a) $x^2 = 4$
   - (b) $|x| = 2$
   - (c) $y^2 = -4$
   - (d) $x^2 < 4$
   - (e) $|x| < 2$
   - (f) $(x + 2)^2 = 9$
   - (g) $(y - 3)^2 = 9$
   - (h) $(x + 2)(x - 3) = 0$
   - (i) $(x + 2)^2 < 5$
   - (j) $(y - 2)^2 < 5$
   - (k) $(x^2) + (3x) = 0$

5. Picture the solution set for each of the exercises in 4 by using a number line and enlarging dots. Thus, if your solution set is $\{-1,2\}$ its picture or graph is:



*6. Using the methods of Section 6.7 prove the following:

(a) $3 \cdot 0 = 0$

(b) $3 \cdot (-2) = -6$

(c) $(-3) \cdot (-2) = 6$

*7. Prove that if the product of two integers is zero, then one of the factors is zero. (Hint: What are the possible signs of a product?)

*8. If $\underline{r}$, $\underline{s}$ and $\underline{t}$ are integers with $r < s$, which of the following statements are true and which are false?

(a) $2r < 2s$ (d) $rt > st$ when $t < 0$

(b) $-2r < -2s$ (e) $r^2 > 0$ when $r \neq 0$

(c) $rt < st$ when $0 < t$ (f) $r + t < s + t$

9. Write equations for each of the following sentences, and then find all integer solutions.

(a) The double of an integer is -12.

(b) The double of an integer is three less than the integer.

(c) The square of an integer is less than 20 and greater than 4.

(d) The sum of an integer and its successor is -7.

(e) The product of an integer and its successor is 42.

*10. Using the distributive property of multiplication over addition, prove: (x is an integer)

(a) $(2x) + (3x) = 5x$

(b) $(2x) + x = 3x$

(c) $(5x) - (2x) = 3x$

(d) $(2x) - (5x) = -3x$

(e) $(5x) - x = 4x$

(f) $x - (5x) = -4x$

11. Solve for x from among the integers.

(a) $(2x) + (3x) = 20$

(b) $(2x) + (3x) = -20$

(c) $3x = 20 + (2x)$

(d) $3x = 20 - (2x)$

(e) $3x = (2x) - 20$

(f) $2x = (3x) + 20$

(g) $20 = (2x) - (3x)$

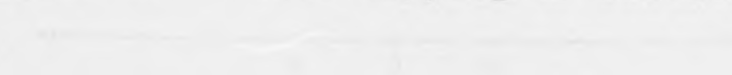
(h) $|2x + x| < 7$

(i) $|2x + 3| + |x - 1| < 10$

6.9 Dilations and Multiplication of Integers

In Chapter 4 we found that the positive and negative integers could be interpreted as translations of the number line to the right or left. Addition of integers was found to correspond closely to composition of translations. Multiplication of integers has a different interpretation on the number line.

Let us begin with a line in which one fixed point is labeled "C." Consider the following mapping of the line onto itself: The mapping assigns point C to itself but to any other point P on the line it assigns the point P' such that P is the midpoint of segment CP'. This mapping is illustrated by the arrow diagram



For this mapping, the distance CP' is twice the distance CP. Thus the mapping corresponds to

$$n \longrightarrow 2n$$

which takes whole numbers into their doubles. If we denote this mapping, which doubles distances from *C*, by "2' " (read: 2 prime), we have

$$2': \quad P \longrightarrow P'$$

$$2': \quad C \longrightarrow C$$

$$2': \quad Q \longrightarrow Q'$$

(In the notation of Chapter 3, this would be written $P \xrightarrow{2'} P'$, etc.)

In a similar manner we define 3' to be the mapping that takes any point *P* into a point that is three times as far from *C*, and on the same side of *C* as *P*. In general, if *d* is any positive integer, "d' " will denote the following mapping:

$$d': \quad C \longrightarrow C,$$

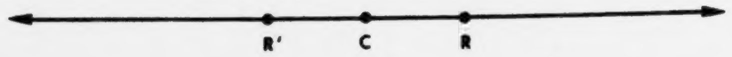
and if *P* is a point on the line distinct from *C*, then

$$d': \quad P \longrightarrow P',$$

where CP = d · CP, and *P* is between *C* and *P'*.

This mapping is called a *dilation* with *center C*.

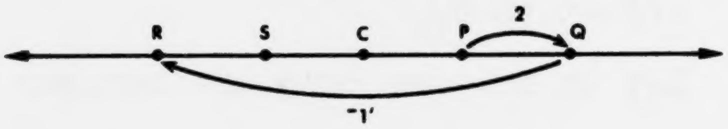
Let us now define another mapping that also leaves *C* fixed. This mapping takes any point *R* to a point on the other side of *C*, the same distance from *C*.



Since this mapping reflects R in C, it is called the *reflection in C* and is denoted "-1' " (read: negative one prime). Such a mapping is also called a symmetry in point C because points R and R' are located symmetrically on either side of C. However, in this chapter we shall continue to call such a mapping a reflection in a point.

$$-1': \; R \longrightarrow R'$$
$$-1': \; R' \longrightarrow R$$
$$-1': \; C \longrightarrow C$$

Let us now see what happens when we compose -1' with 2'. Both mappings leave C fixed, so let point P be different from point C. Locate points Q, R, and S so that RS = SC = CP = PQ, and the points are in the indicated order.



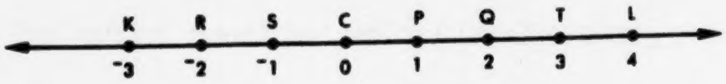
Then $2': \; P \longrightarrow Q$ and $-1': \; Q \longrightarrow R$. The composition of -1' with 2' takes P into R via Q. Similarly, the composition of 2' with -1' takes P into R via S.

We shall see that composition of such mappings is analogous to multiplication of integers. Anticipating this analogy, let us agree to express this composition by use of the multiplication sign "x." We may now write

$$(2' \times (-1')): \; P \longrightarrow R.$$
$$((-1') \times 2'): \; P \longrightarrow R.$$

We shall use "-2'" as an abbreviation for "2' x -1'." Similarly, -3' = 3' x -1' and -4' = 4' x -1'. We shall also say that -2' contains a reflection. -3', -4', -5', ... also are said to contain a reflection.

Let us look at a few more examples. (We take the point labeled "0" on the number line as the center.)



Example 1. $3': P \longrightarrow T$ or $1 \longrightarrow 3$

$-3': P \longrightarrow K$ or $1 \longrightarrow -3$

Example 2. $3': S \longrightarrow K$ or $-1 \longrightarrow -3$

$-3': S \longrightarrow T$ or $-1 \longrightarrow 3$

Example 3. $((-2') \times 2'): S \longrightarrow L$ or $-1 \longrightarrow 4$

because $2': S \longrightarrow R$ and $-2': R \longrightarrow L$.

Note that $-4': S \longrightarrow L$ or $-1 \longrightarrow 4$ so that the mappings $-2' \times 2'$ and $-4'$ have the same effect on S.

Example 4. Let us now use only the integer names for the points.

$$(2' \times 3'): 1 \longrightarrow 6, \quad -4 \longrightarrow -24$$
$$6': 1 \longrightarrow 6, \quad -4 \longrightarrow -24$$
$$((-2') \times 3'): 1 \longrightarrow -6, \quad -4 \longrightarrow 24$$
$$-6': 1 \longrightarrow -6, \quad -4 \longrightarrow 24$$
$$((-2') \times (-3')): 1 \longrightarrow 6, \quad -4 \longrightarrow -24$$

What do these examples suggest?

It will be convenient to define the magnitude of such a dilation mapping. The magnitude of the mapping $\underline{d}'$, where "d" names any integer, is the same as the absolute value of $\underline{d}$, that is $|d|$. We shall use the same vertical bar notation to denote magnitude. Thus, $|d'| = |d|$. In particular

$$|3'| = |3| = 3$$
$$|-3'| = |-3| = 3$$

Let r and s be any integers, r' and s' their corresponding mappings. Then the composite mapping r' x s' has the following property:

$$|r' \times s'| = |r'| \cdot |s'|$$

because s' enlarges by a factor of $|s'|$ and r' enlarges the enlargement by a factor of $|r'|$. The net result is to enlarge by a factor of $|r'| \cdot |s'|$.

If neither r' nor s' contains a reflection, the composition mapping r' x s' contains no reflection. If both r' and s' contain reflections, then r' x s' contains no reflection. If either r' or s' (but not both) contains a reflection, then r' x s' contains a reflection. Let us say that r' and s' have the same direction if either both contain reflections or neither contains a reflection. Then r' and s' are the same mapping if they have the same direction and magnitude.

Let us call every mapping d', where d is an integer, a dilation. The point "C" that d' maps into itself is called the center of the dilation. The set of dilations with center C, together with the operation "x" expressing compositions determine a mathematical system which we shall denote "(D',x)."

To compute the composition of two mappings will mean to express the composite mapping as a mapping without an indicated composition. Thus, the computed mapping for (-3') x (-2') is 6' and we shall write (-3') x (-2') = 6' because (-3') x (-2') and 6' have the same direction and magnitude.

The resemblance between (Z,·) and (D',x) should be quite apparent by now. In the first place, there is a one-to-one cor—

respondence between the integers Z and the dilations D'. Moreover, composition of dilations strongly suggests how we should multiply integers.

6.10 Exercises

Use the integer names for points of our number line and let our dilations be with center 0.

1. Into what points does $7'$ map each of the following?

   (a) 6 (b) -6 (c) 1 (d) 0

2. Into what point does $-7'$ map each of the following?

   (a) 6 (b) -6 (c) 1 (d) 0

3. Into what point does $2' \times -3'$ map each of the following?

   (a) 1 (b) -1 (c) 0 (d) 10 (e) -10

4. Compute:

   (a) $(-7') \times (-6')$

   (b) $7' \times (-6')$

   (c) $6' \times (-7')$

   (d) $(-6') \times 7'$

   (e) $(-15') \times (-15')$

   (f) $25' \times 25'$

   (g) $(-35') \times (-35')$

   (h) $45' \times 45'$

   (i) $(2' \times (-3')) \times 4'$

   (j) $2' \times ((-3') \times 4')$

   (k) $((-17') \times 25') \times 4'$

   (l) $(-17') \times (25' \times 4')$

*5. Let $\underline{r}'$, $\underline{s}'$, $\underline{t}'$ be any dilations. Prove that composition of dilations

   (a) is commutative: $r' \times s' = s' \times r'$

   (b) is associative: $(r' \times s') \times t' = r' \times (s' \times t')$

   (c) has $1'$ as an identity.

6. What can you say about the dilation $0'$? Is it a mapping?

7. Is $(D', \times)$ an operational system?

8. Can a dilation have more than one fixed point? Explain your answer?

6.11 Summary

1. In this chapter we developed and studied multiplication of integers from various points of view. The definition of the multiplication operation was motivated by the desire to extend the close connection between (W,+) and (Z,+), to maintain patterns previously known to hold for multiplication in W, and to preserve certain nice properties of (W,+,.). Multiplication received further interpretation as a composition of dilation mappings.

2. For any integers r and s, the product r · s is
   (1) 0 if r or s is zero;
   (2) positive if both r and s are positive or both r and s are negative;
   (3) negative if one is positive and the other negative.

   Furthermore, the absolute value of r · s is |r| · |s|.

3. If multiplication of integers is assumed to be commutative, associative, and distributive, then the definition of the product r · s in the preceding paragraph is the only possible one.

6.12 Review Exercises

1. Compute:

(a) $9 + (-7)$
(b) $9 - (-7)$
(c) $(-9) - (-7)$
(d) $(-9) \cdot (-7)$
(e) $9 \cdot (-7)$
(f) $(-12)^2$
(g) $|(-23) \cdot 9|$
(h) $|-23| \cdot |9|$
(i) $(-47) \cdot (17 - 25)$
(j) $(39 \times (-27)) - (39 \times (-17))$
(k) $(29 \times (-7)) \times (29 \times (-13))$
(l) $(47^2) - (48^2)$

2. Find the solution set from the set of integers.

(a) $x^2 = 9$
(b) $y^2 - 1 = 0$
(c) $(-2)s = 8$
(d) $r^2 < 5$
(e) $x^2 = -1$
(f) $x(x + 2) = 0$
(g) $n(n + 1) = 55$
(h) $(x + 1)^2 = 4$
(i) $|r^3| < 100$
(j) $s^2 = -s$

3. Picture on a number line the solution set for each exercise in 2.

4. Answer TRUE (T) or FALSE (F).

(a) Multiplication of integers is both commutative and and associative.

(b) Multiplication of integers distributes over both addition and subtraction.

(c) Multiplication of an integer by -2 always gives a smaller integer.

(d) Subtraction of integers is associative.

(e) If a product of integers is 0, one of its factors must be 0.

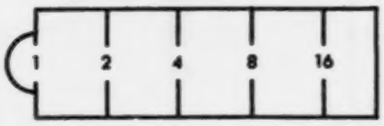
(f) If a product of integers is negative, then at least one of the factors must be negative.

(g) If r, s, t are integers and $(rs)t < 0$ then r or s or t must be positive.

(h) If one of the factors of a product of integers is 0 then the product is 0.

(i) In $(Z,\cdot)$ if a product is 0 then one of the factors must be 0.

5. Make two strips with scales as shown:



(a) Try to find a way of using your strips to compute products. Draw a picture showing the position of your strips for the products

(1) 2 x 2
(2) 2 x 4
(3) 2 x 8
(4) 4 x 2
(5) 4 x 4

(b) Notice that the scales do not show all the whole numbers. Should the exact midpoint between the markings for 2 and 4 be 3? If not, should it be more or less? Why do you think so? The strips you have constructed make a crude slide rule for multiplication.

*6. The figure shown is a *nomogram* for multiplication. The figure shows how to compute 2 x 4. Draw lines to show the computation for

(a) 2×2

(b) 2×4

(c) 2×8

(d) $2 \times \frac{1}{2}$

(e) 4×4

(f) 4×8

(g) 8×8

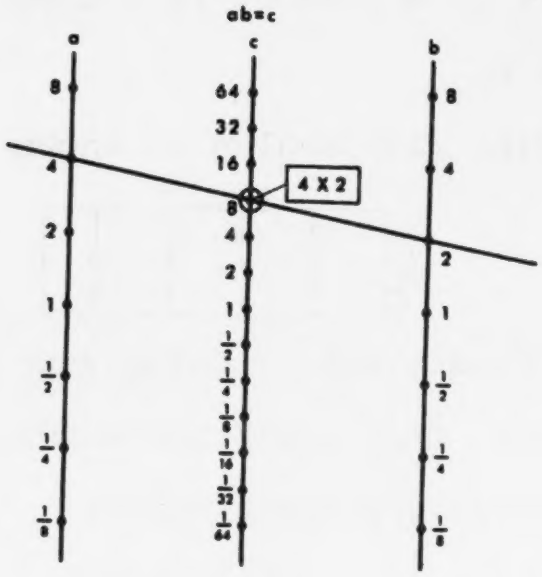
(h) $\frac{1}{2} \times 8$

(i) $\frac{1}{8} \times \frac{1}{2}$

(j) $8 \times \frac{1}{4}$

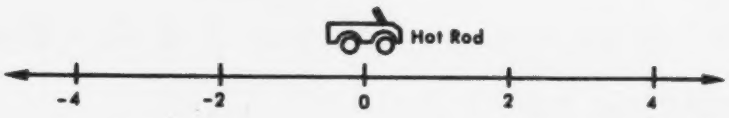
(k) $\frac{1}{4} \times \frac{1}{4}$

(l) $\frac{1}{8} \times \frac{1}{8}$


ab=c
a
c
b
8
4
2
1
1/2
1/4
1/8
64
32
16
8
4
2
1
1/2
1/4
1/8
1/16
1/32
1/64
8
4
2
1
1/2
1/4
1/8
4 X 2

7. (a) If a hot rod moves at a fixed speed of 4 feet per second to the right, what interpretation would you give to a speed of -4 feet per second?

(b) If the hot rod starts at 0 on the number line (measured in units of 1 foot) and has a speed of 4 feet per second (fps), where will the hot rod be in 3 seconds? If we think of the place on the number line the hot rod is at the moment, how might we interpret the instant -3 seconds?


Hot Rod
-4
-2
0
2
4

(c) Let us agree to interpret 4 x 3 as a product that locates the hot rod on the number line if it starts at 0, where 4 is the speed in fps, and 3 is the number of seconds from the time it was at 0. Interpret the following products and see whether your interpretations are consistent with our rules for multiplying integers:

(1) 4 x -2
(2) -4 x 2
(3) -4 x -2

CHAPTER 7

LATTICE POINTS IN THE PLANE

7.1 Lattice Points and Ordered Pairs

The word "lattice" in the title of this chapter suggests an open network like a trellis, and indeed that is its origin. You probably heard about the geodesic dome, designed by R. Buckminster Fuller, that is large enough to house a baseball park. If this dome were flattened, part of it would look like Figure 7.1.

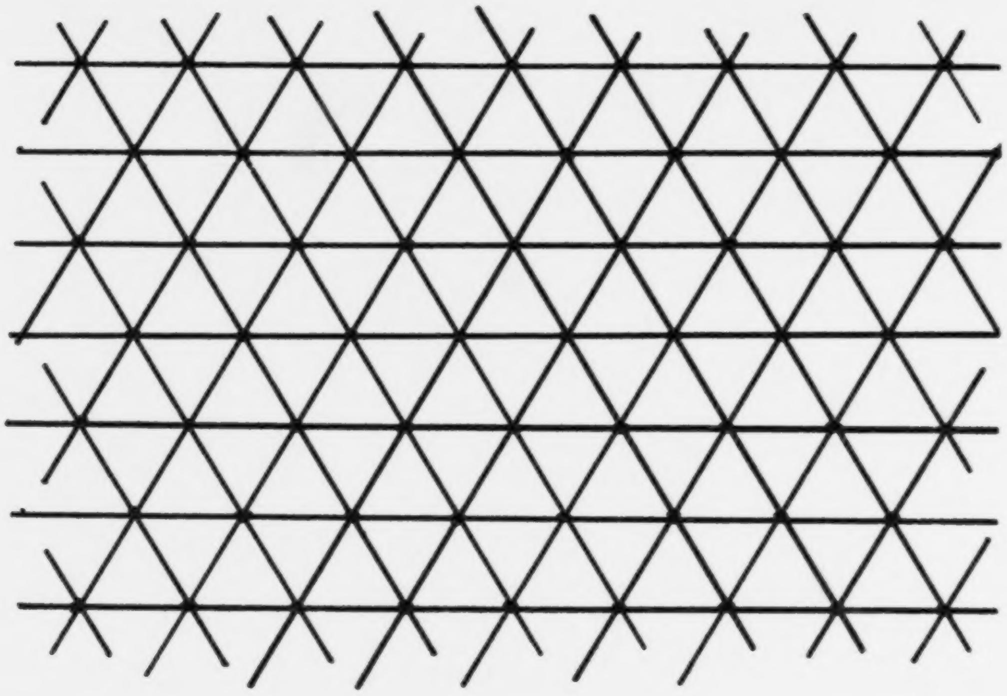


Figure 7.1

The set of intersection points above illustrates a lattice. We use lattices in this chapter to help us understand mappings better.

As you can see, Figure 7.1 seems to be built up with triangles. However if we remove one set of parallel lines, we see parallelograms instead of triangles. But we have the same lattice. This is shown in Figure 7.2 below:

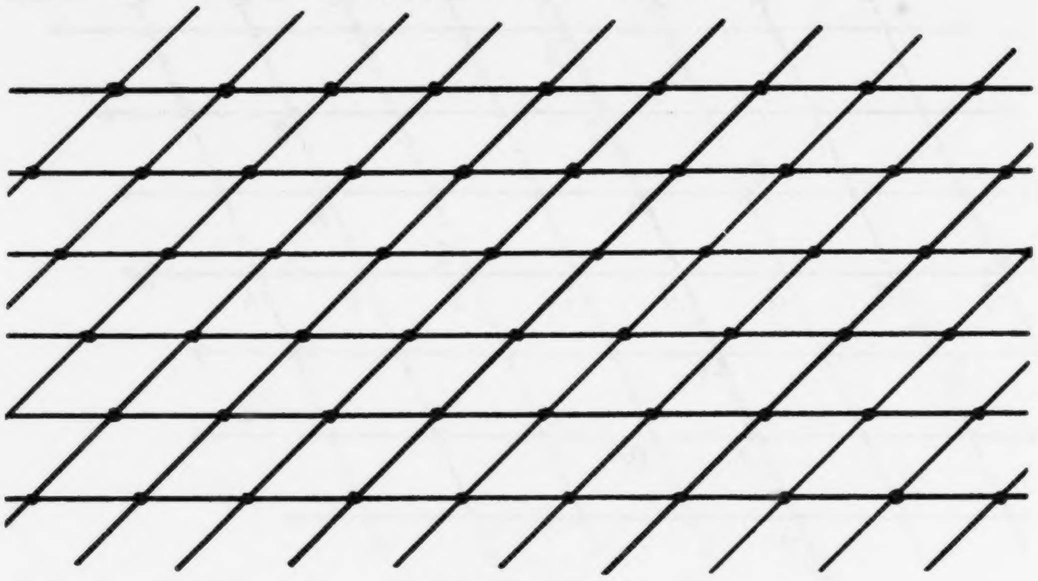


Figure 7.2

Note the following features of a lattice.

1. The points of a lattice are determined by two families or sets of parallel lines, each line in one family intersecting every line in the other. This implies that all lines are in one plane.
2. The lines in each family are evenly spaced. But the spacing for one family need not be the same as the spacing for the other.
3. Each lattice point is on two lines, one from each of the two families of lines.

These features suggest a method of using integers to describe precisely the location of lattice points. We start by choosing two lines, one from each family, calling one the x-axis and the other the y-axis. Then we assign integers to the lattice points on these axes, in the same manner as we would to a number line, reserving zero for the point of intersection of the axes. Figure 7.3 shows the start of this.

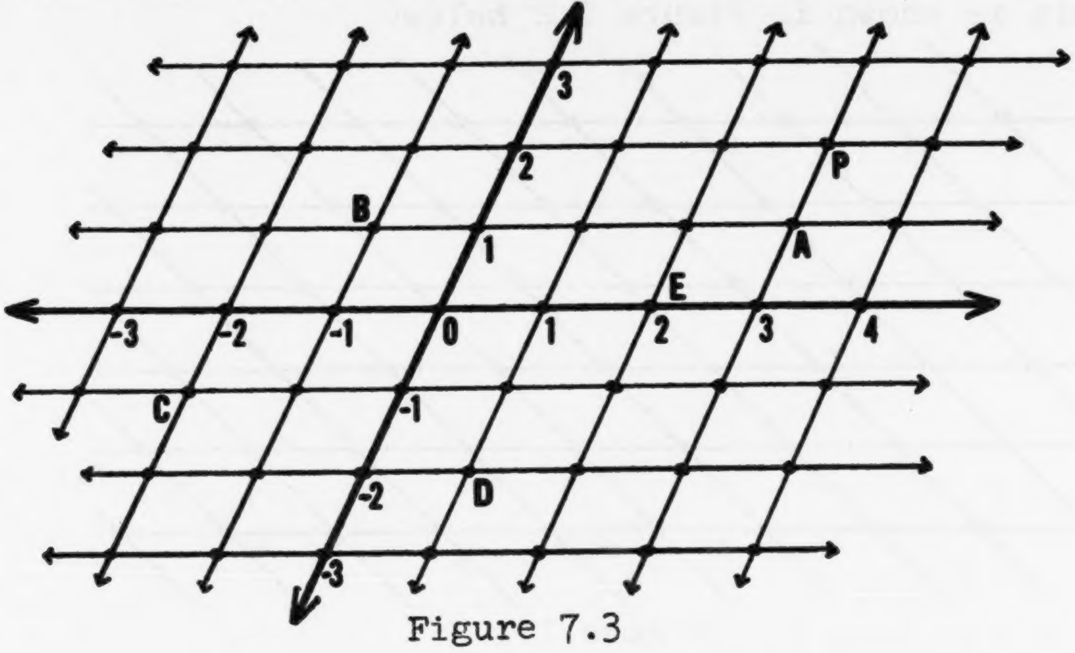


Figure 7.3

With this foundation we can assign an ordered pair of integers to any lattice point. We illustrate how this is done for the point named P in Figure 7.3. Point P lies on two lines. One cuts the x-axis at the point whose assigned integer is 3. The other cuts the y-axis at the point whose assigned integer is 2. Taken in that order, the pair of integers assigned to P is 3,2, which we write (3,2). The parentheses, as you know, indicate an ordered pair, the first integer is called the x-coordinate of P; the second is called the y-coordinate of P; and together they are called the coordinates of P.

Note that the arrow heads in Figure 7.3 indicate that the lattice extends over the entire plane. For this reason we have need for all the integers.

A system that assigns ordered pairs of numbers to points in a plane is called a plane coordinate system. The system we have described assigns ordered pairs of integers to a set of lattice points in a plane. We can call our system a plane lattice coordinate system.

The set of ordered pairs of integers in the plane lattice coordinate system is often referred to as Z X Z, read "Z cross Z." The Z is the same symbol as we used for the set of integers. The X in Z X Z suggests ordered pairs.

In the diagrams below we show a variety of lattice coordinate systems. Study them and see how they differ.

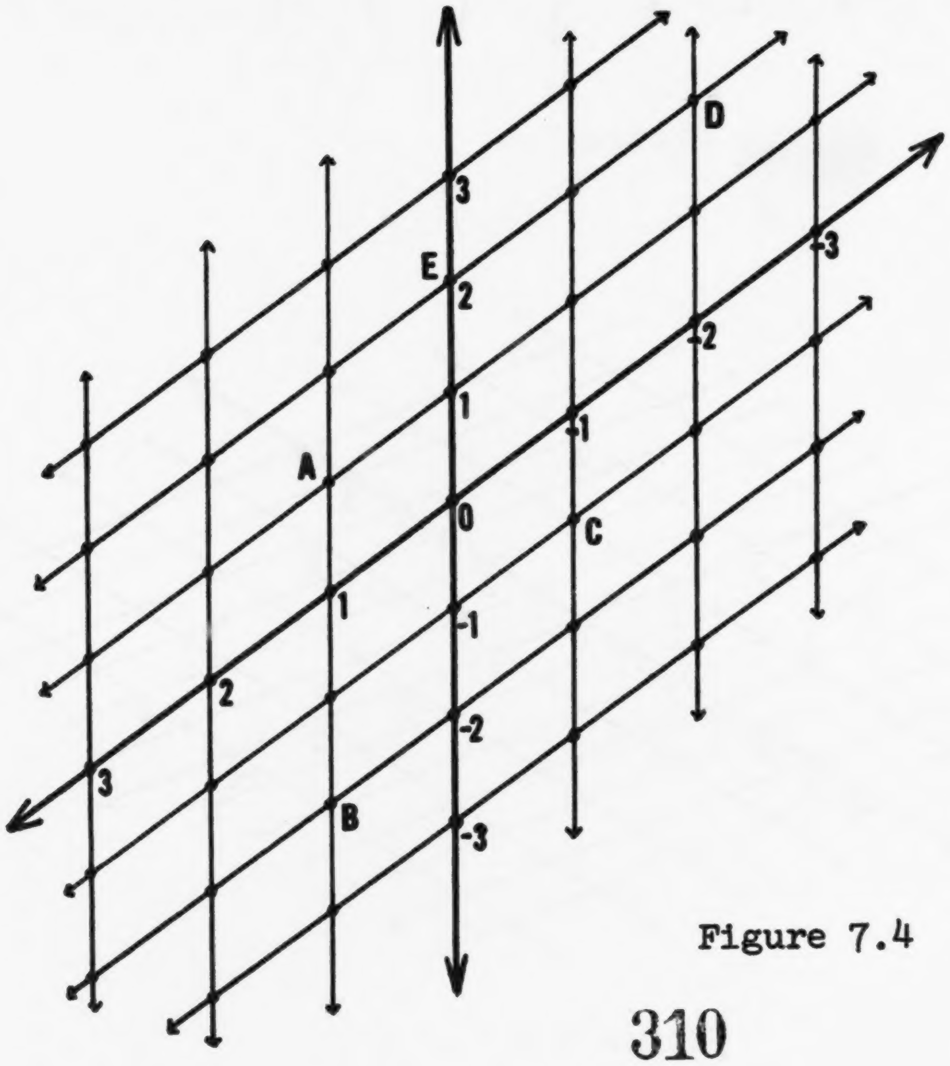


Figure 7.4

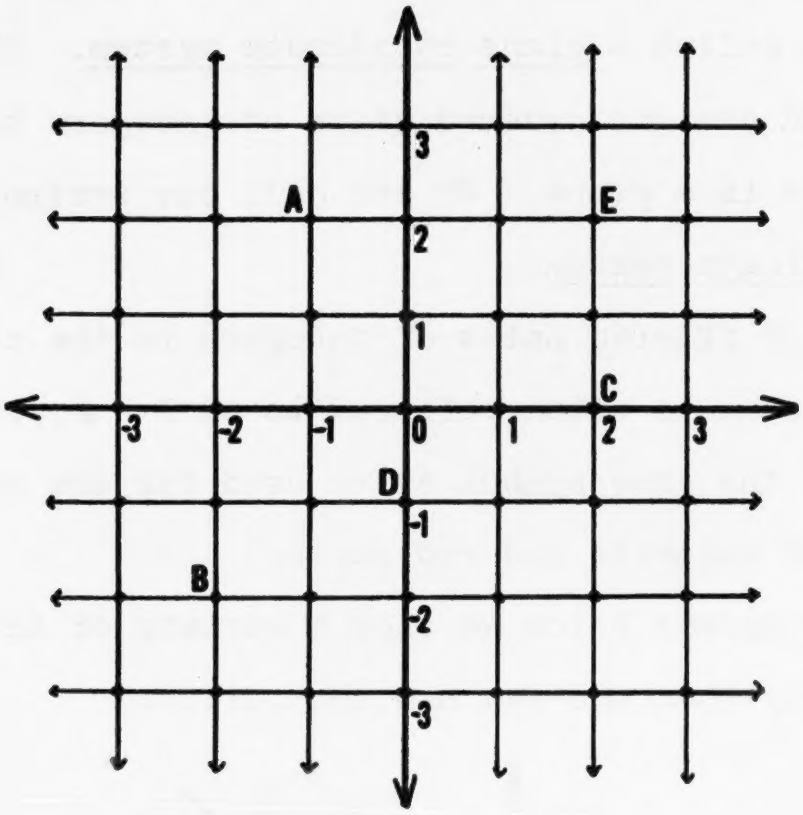

3
2
1
0
-1
-2
-3
-3
-2
-1
1
2
3
A
B
C
D
E

Figure 7.5

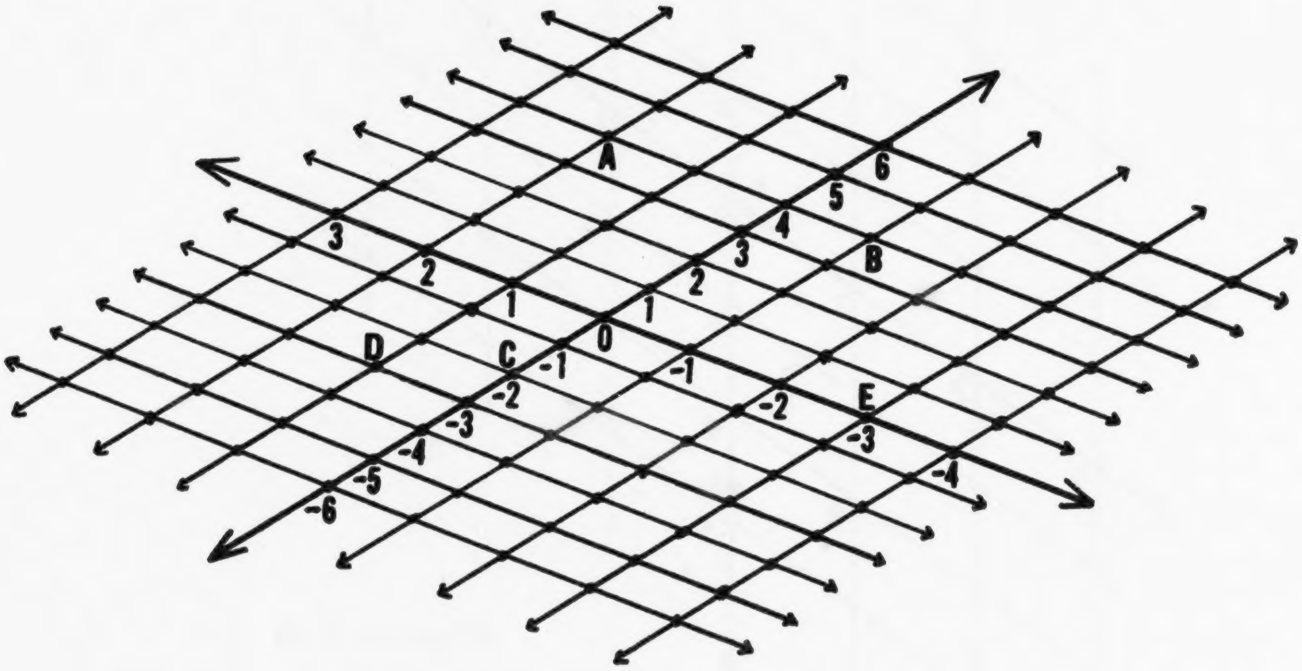

A
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
3
2
1
-1
-2
-3
-4
B
C
D
E

Figure 7.6

We have seen that, given a lattice coordinate system and a point in the lattice, we can assign an ordered pair of integers to the point. Can we reverse the assignment? That is, given a lattice coordinate system and an ordered pair of integers, can we assign a point of the lattice to the ordered pair? Let us see. Suppose the pair of integers is (-2, -1). We start by finding the point on the x-axis whose assigned integer is -2. There is exactly one line parallel to the x-axis through this point. Then we find the point on the y-axis whose assigned integer is -1. There is exactly one line parallel to the x-axis through this point. These lines, belonging to different families of parallel lines, intersect at exactly one point, and this is the point whose coordinates are (-2, -1).

Using the lattice coordinate system in Figure 7.3, locate the point whose coordinates are (-2, -1). Locate the point in the lattice coordinate system in Figure 7.4 whose coordinates are (-2, -1). Repeat for Figures 7.5 and 7.6.

7.2 Exercises

1. Find the coordinates of the points named A, B, C, D, E in

   (a) Figure 7.3 (c) Figure 7.5

   (b) Figure 7.4 (d) Figure 7.6

2. In a lattice coordinate system, is there a lattice point with coordinates

   (a) (300, 282)? (b) (-5062, -4)? (c) ($2\frac{1}{2}$, 0)?

   For the exercises that follow you will need some lattice paper (perhaps your teacher will have a supply dittoed), some colored pencils, and a ruler. The lattice paper should have at least eleven rows of dots and eleven columns of dots. Draw a line through a row of dots to serve as the x-axis and a line through a column of dots to serve as the y-axis. See Figure 7.3. Ordinary graph paper can also be used.

3. Locate on a lattice coordinate system the points that have the following coordinates:
    (a) (3,4)
    (b) (-3,4)
    (c) (3,-4)
    (d) (-3,-4)
    (e) (1,0)
    (f) (0,1)
    (g) (-2,0)
    (h) (0,-2)
    (i) (-5,6)
    (j) (6,-5).

4. Select the seven consecutive points on the x-axis whose middle point has coordinates (2,0). What are the coordinates of the other six points?

5. Select five consecutive points on the y-axis whose middle point has coordinates (0,-2). What are the coordinates of the other four points?

6. Draw a line or lines with colored pencil through the points whose coordinates satisfy the following conditions. Use a different color for each condition in a group and a different sheet of lattice paper for each group.

    Group 1:
    (a) The first coordinate is equal to the second coordinate.
    (b) The first coordinate is the additive inverse of the second.

Group 2:

(c) The sum of the coordinates of the point is 5.

(d) The sum of the coordinates is 3.

(e) The sum of the coordinates is -3.

(f) The sum of the coordinates is -5.

Group 3:

(g) The first coordinate minus the second is 2.

(h) The first coordinate minus the second is -1.

Group 4:

(i) The first coordinate equals 2.

(j) The first coordinate equals -2.

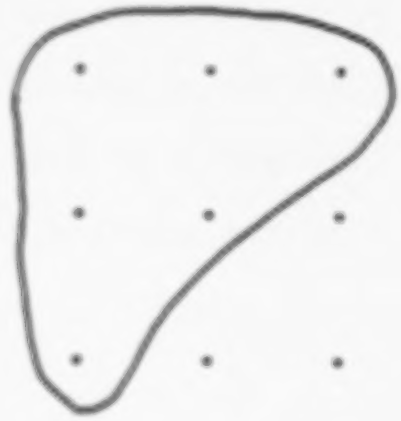
(k) The second coordinate equals 4.

(l) The second coordinate equals -4.

Group 5:

(m) The absolute values of the coordinates are equal.

7. For each condition listed in this exercise use a different color to draw a closed curve enclosing just those points, represented on your graph or lattice paper, that satisfy the condition. For example:



Note: Enclose only the points on your lattice paper, even if there are points not on your lattice paper which satisfy the condition.

(a) The first coordinate is less than the second.
(b) The first coordinate is greater than the second.
(c) The sum of the coordinates is greater than 5.
(d) The sum of the coordinates is less than -5.
(e) The first coordinate is less than -2.
(f) The first coordinate is greater than 3.
(g) The second coordinate is less than -4.
(h) The second coordinate is greater than 3.

7.3 Conditions on Z X Z and their Graphs

The set of ordered pairs that satisfies one of the conditions in Exercise 6 or 7 in Section 7.2 is called the solution set of that condition. For example, the solution set of the condition "The sum of the coordinates is five," would include

(0,5), (1,4), (2,3), (3,2), (4,1), (5,0), (6,-1),
(7,-2), ..., (-1,6), (-2,7), (-3,8), ...

The set of lattice points associated with these ordered pairs is called the graph of the solution set, or sometimes the graph of the condition. The graph of the above solution set is represented in Figure 7.7 by the circled points in the lattice.

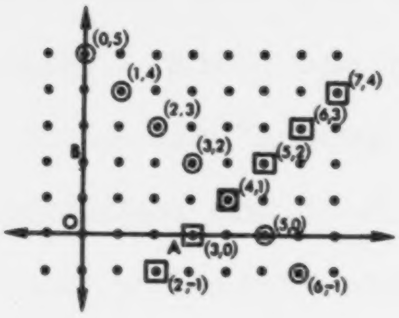


Figure 7.7

Notice that in Figure 7.7 the graph of the condition "The first coordinate is 3 more than the second coordinate" is displayed by enclosing the points in squares. (Very often it is effective to display the graphs of different conditions by using different colors to enclose the points.)

Questions: Which point is enclosed by both a circle and a square? Is 4 + 1 equal to 5? Is 4 three more than 1? Does (4,1) satisfy both conditions?

Part of our study of mathematics is learning to express mathematical ideas in the symbolism of mathematics. You have previously used "x" to express the first coordinate and "y" to express the second.

Therefore, instead of writing "The sum of the coordinates is 5," we can write "$x + y = 5$." Instead of writing "The first coordinate is 3 more than the second coordinate," we can write "$x = y + 3$." If we are interested in the pair of numbers that satisfies both of those conditions, we can write, "$x + y = 5$

and $x = y + 3$." This new condition is made up of two conditions connected by "and." Its solution set is $\{(4,1)\}$ and the graph is a set containing only one point. This point is called the intersection of the two graphs (the graph of the condition "$x + y = 5$" and the graph of the condition "$x = y + 3$"), and $\{(4,1)\}$ is called the intersection of the two solution sets. The sentences that we write to represent conditions are "open sentences." (See 1.7).

7.4 Exercises

1. Translate the following conditions to the forms used above, making use of the symbols "x," "y," "$=$," etc.
   - (a) The first coordinate is equal to the second coordinate. (Ans. $x = y$)
   - (b) The first coordinate is the additive inverse of the second coordinate. (Ans. $x = -y$)
   - (c) The sum of the coordinates is three.
   - (d) The sum of the coordinates is -3.
   - (e) The sum of the coordinates is -5.
   - (f) The difference of the first and second coordinates (in that order) is 2.
   - (g) The difference of the first and second coordinates is -1.
   - (h) The first coordinate equals 2.
   - (i) The first coordinate equals -2.
   - (j) The second coordinate equals 4.

(k) The absolute values of the coordinates are equal.

2. Draw the graphs for the open sentences you wrote in Exercise 1.

3. Translate the following conditions into words (in terms of coordinates):

   (a) $x + 6 = y$
   (b) $y - x = 3$
   (c) $y = |x|$
   (d) $y = x - 2$
   (e) $7 = |x - 3|$
   (f) $x = 7$
   (g) $y = 1$

4. Using ">" for "greater than" and "<" for "less than," translate the sentences of Section 7.2, Exercise 7 into mathematical symbols.

5. Translate the following into mathematical symbols:

   (a) The second coordinate is the product of 2 and the first coordinate.
   (b) The first coordinate is the product of 2 and the second coordinate.
   (c) The second coordinate is the product of 3 and the first coordinate.
   (d) The first coordinate is the product of 3 and the second coordinate.

6. Describe the following conditions in words:

   (a) $y = 5x$
   (b) $x = 5y$
   (c) $y = x^2$
   (d) $y = 0$
   (e) $y < 0$
   (f) $x > 0$
   (g) $x \cdot y = 6$
   (h) $2x = 3y$

7. For each of the conditions in Exercise 6, list four members of Z X Z that satisfy the condition. For example, (1,5), (2,10), (-1,-5) and (0,0) are four members of Z X Z that

satisfy 6(a).

8. Use the same sheet of lattice paper to graph each of the following conditions. Use a different color for each condition to circle the points that satisfy the condition.

   (a) $y = x$ (c) $x = 27$ (e) $y = 0$

   (b) $y = 2x$ (d) $x = 0$

9. What is the intersection (common point) of the graphs in Exercise 8? Which graph was included in the x-axis? the y-axis? Which of the graphs were contained in a line other than an axis?

10. Translate the following into mathematical symbols:

    (a) The second coordinate is 1 more than twice the first coordinate.

    (b) The first coordinate is 5 less than 3 times the second coordinate.

11. Describe the following conditions in words.

    (a) $y = x + 1$ (c) $y = x + 2$

    (b) $y = x - 1$ (d) $y = x - 2$

12. For each condition in Exercise 11, draw a line through the points that satisfy the condition. Use the same sheet of lattice paper for all lines.

13. In what way were the four lines in Exercise 12 alike? List the coordinates of the points in which the lines intersected the y-axis. Note the similarity between these coordinates and the conditions as expressed in Exercise 11.

7.5 Intersection and Unions of Solution Sets

All lattice points satisfying the condition $x > 0$ are located on the same side of the y-axis. We will designate the set of lattice points on this side of the y-axis by "A." The set of lattice points satisfying the condition $y > 0$ is located on the same side of the x-axis. Call this set "B."

When two conditions are joined by a connective such as "and" they form a new condition called a compound condition. The set of points which satisfy the compound condition "$x > 0$ and $y > 0$" is the set which satisfies both the conditions "$x > 0$" and "$y > 0$." This set is called the intersection of sets A and B since it consists of all those elements that are in both A and B. Figure 7.8 illustrates the relationship of sets A,B and the intersection of A and B (written $A \cap B$).

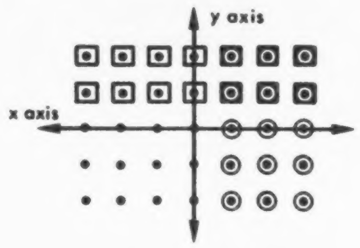


Figure 7.8

Points in A are in circles. ($x > 0$)

Points in B are in squares. ($y > 0$)

Points in $A \cap B$ are in circles and squares.

($x > 0$ and $y > 0$)

Let C be the set of points satisfying the condition "$x < 0$."

Let D be the set of points satisfying the condition "$y < 0$." Illustrate C,D and $C \cap D$ in a diagram such as Figure 7.8. Repeat the preceding instructions for A and D, then for B and C.

List the coordinates for two points in (1) $A \cap B$ (2) $C \cap D$ (3) $A \cap D$ (4) $B \cap C$.

All the lattice points satisfying the condition "$x = 0$" are on the y-axis. Call this set "E." The solution set of the compound condition "$x > 0$ or $x = 0$" contains those "points" which satisfy either "$x > 0$" or "$x = 0$" or both. This set is the union of A and E, written $A \cup E$. Figure 7.9 illustrates this set relationship:

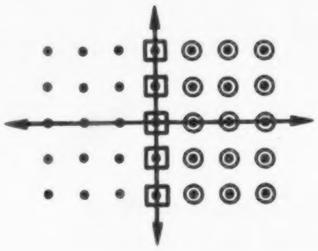


Figure 7.9

Points in A are enclosed by circles. ($x > 0$)

Points in E are enclosed by squares. ($x = 0$)

Points in $A \cup E$ are enclosed. ($x > 0$ or $x = 0$)

A simpler notation for "$x > 0$ or $x = 0$" is "$x \geq 0$" and is read "x is greater than or equal to zero."

7.6 Exercises

1. In this exercise try to locate the points in the graph of the compound conditions without first graphing each simple condition separately. Do all parts of this exercise on one sheet of lattice paper.

   (a) $x \geq 0$ and $x = y$.

   (b) $x < 0$ and $x = -y$.

   (c) ($x \geq 0$ and $x = y$) or ($x < 0$ and $x = -y$).

   In Exercises 2, 3, 4 and 5, follow the instructions of Exercise 1.

2. (a) $x \geq -1$ and $y = x + 1$.

   (b) $x < -1$ and $y = -(x + 1)$.

   (c) ($x \geq -1$ and $y = x + 1$) or ($x < -1$ and $y = -(x + 1)$.

3. (a) $x \geq 0$ and $y \geq 0$ and $x + y = 5$.

   (b) $x < 0$ and $y \geq 0$ and $y - x = 5$.

   (c) ($x \geq 0$ and $y \geq 0$ and $x + y = 5$) or
   ($x < 0$ and $y \geq 0$ and and $y - x = 5$).

4. (a) $x \leq 0$ and $y \leq 0$ and $x + y = -5$.

   (b) $x \geq 0$ and $y \leq 0$ and $x - y = 5$.

   (c) ($x \leq 0$ and $y \leq 0$ and $x + y = -5$) or
   ($x \geq 0$ and $y \leq 0$ and $x - y = 5$).

5. (a) $y \geq x$ and $y \leq x + 3$.

   (b) $y \leq x$ and $y \geq x - 3$.

   (c) ($y \geq x$ and $y \leq x + 3$) or ($y \leq x$ and $y \geq x - 3$).

7.7 Absolute Value Conditions

In Chapters 4 and 6 you thought of the absolute value of an integer a as max (a,-a). From this definition you can see that:

(a) The absolute value of zero is zero.

(b) The absolute value of a positive integer is that positive integer.

(c) The absolute value of a negative integer is the additive inverse of that negative integer.

This covers all possibilities since $x = 0$, $x > 0$ or $x < 0$, if x is an integer.

A more compact way of writing this definition is:

$$|x| = \begin{matrix} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{matrix}$$

Example 1: If $x = 5$, $|x| = 5$ since $5 > 0$.

If $x = 0$, $|x| = 0$ since $0 = 0$.

If $x = -3$, $|x| = 3$ since $-3 < 0$ and $-(-3) = 3$.

Example 2: Suppose $|x| = 3$.

From the definition $|x| = x$ or $|x| = -x$, therefore, substituting 3 for $|x|$ in the line above, $3 = x$ or $3 = -x$ which implies $x = 3$ or $x = -3$.

You see that we started with $|x| = 3$, and found as a result the compound condition "$x = 3$ or $x = -3$." The solution set of this condition is the union of the solution sets of the two simple conditions.

On a line this solution set is simply a pair of points. In the set of lattice points in the plane, however, a more interesting situation develops. In Figure 7.10 the points for which $x = 3$ or $x = -3$ are circled.

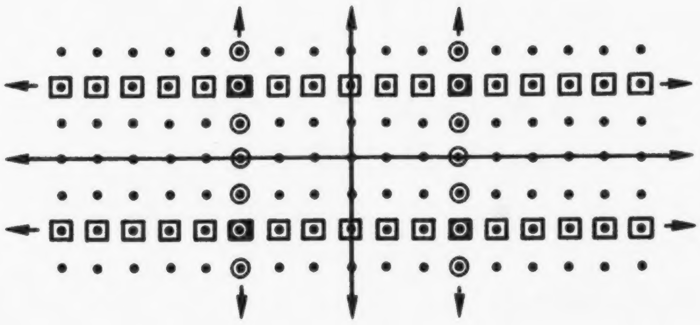


Figure 7.10

Furthermore, suppose $|y| = 2$, then $y = 2$ or $y = -2$. In Figure 7.10 the points for which the second coordinates are 2 or -2 are enclosed in squares. In what way is the graph of $|x| = 3$ and $|y| = 2$ indicated?

7.8 Exercises

1. What integers do the following name?

   (a) $|-7|$ (b) $|15|$ (c) $|0|$ (d) $|-1|$ (e) $|999|$.

2. Graph the following on the same lattice.

   (a) $|x| = 4$

   (b) $|y| = 1$

   (c) $|x| = 4$ and $|y| = 1$.

   (d) $|x| = 4$ or $|y| = 1$.

   (e) Describe how the graphs in (c) and (d) are determined by the graphs in (a) and (b).

3. Draw the graph of $y = |x|$. Remember that if $x \geq 0$ then $y = x$; and if $x < 0$ then $y = -x$. $x \geq 0$ simply states that the points are to the right of the y-axis or on the y-axis. $x < 0$ states that the points are to the left of the y-axis.

4. Draw the graph of $y = |x + 1|$.

Hint:

$$|x + 1| = \begin{array}{l} x + 1, \text{ if } x \geq -1 \\ -(x + 1), \text{ if } x < -1 \end{array}$$

Also see Exercise 3 Section 7.6.

5. Graph the following:

(a) $y = 2|x|$. (To the right of the y-axis this becomes $y = 2x$; to the left, $y = -2x$.)

(b) $y = 3|x|$

(c) $y = -2|x|$

*6. Graph the following:

(a) $y = |x| + 1$ (Why can you think of this as the graph of $y = |x|$ translated one space away from the x-axis?)

(b) $y = |x| - 2$

*7. Graph $|x| + |y| = 5$.

7.9 Lattice Point Games

1. The Game of Caricatures

It is interesting to see what happens to a graph or picture when you change the angle at which the x-axis and y-axis intersect. For example see what happens to "square-head" when you change the angle of the axes:

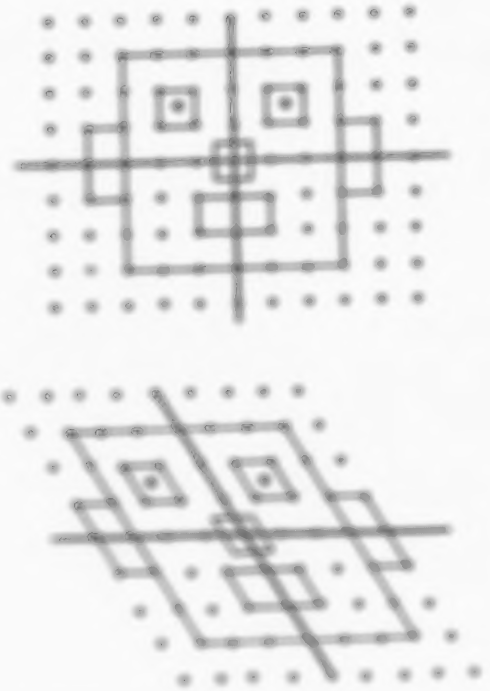


Figure 7.11

What do you think would happen to a circle if you draw it on one grid and then transfer it to another by connecting points with the same coordinates?

Transfer the "man in the moon," pictured in Figure 7.12, to another grid with the axes at a considerably different angle, e.g. +. Use the coordinates of points on the picture to make the transfer.

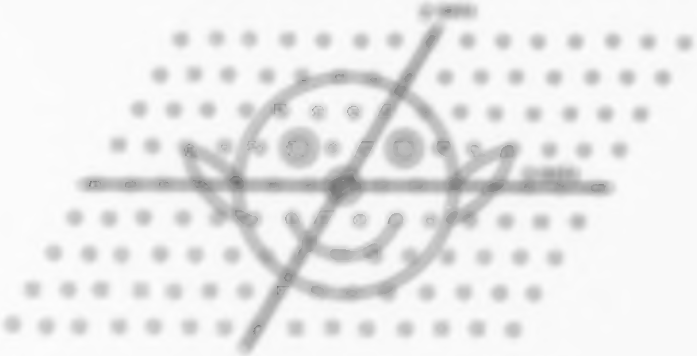


Figure 7.12

Remember that when you find the second coordinate you have to count the points along a "slanted" line. Coordinates for the "man in the moon" are different from that in Figure 7.12.

Head: (-2,4) (2,2) (4,-2) (3,-4) (1,-4) (-2,-2) (-4,2)

Eyes: (-2,2) (0,2)

Nose: (0,0)

Mouth: (-1,-1) (1,-2) (2,-1)

Left Ear: (-4,2) (-5,2) (-4,1)

Right Ear: (2,2) (3,2) (3,1)

To play the game of caricatures:

(a) One student draws a picture on a grid of his own choosing and without showing the picture supplies only the coordinates of key points in the picture.

(b) The other students on self-made grids, using any desired angle for the axes, plot the coordinates on their own grids and sketh in the picture.

*2. Operational Checkers

This game is played by two players on a finite set of lattice points. For example:

(0,2) (1,2) (2,2)

(0,1) (1,1) (2,1)

(0,0) (1,0) (2,0)

You will need to use arithmetic of $(Z_3,*)$ so we will list the necessary facts: 0 * 0 = 0; 0 * 1 = 1; 0 * 2 = 2; 1 * 1 = 2; 1 * 2 = 0; 2 * 2 = 1; and the commutative property will provide the other basic facts.

(a) One player has red checkers and the other has black checkers. A coin is tossed to determine who starts.

(b) The first player places a checker on any point that he wishes.

(c) The second player may then place a checker on any uncovered point and another point with coordinates obtained by adding the corresponding coordinates of the last two points covered. The addition to be used is that for $(Z_3, +)$.

(d) On each subsequent play, if the player's opponent had just placed a checker on (c,d), then the player may not only cover any uncovered point (a,b) but also $(a + c, b + d)$. If this point is already covered by his opponent's checkers, the player replaces it with one of his own. For example, if one player has just covered $(2,1)$, the other player may cover $(2,2)$ and also $(2 + 2, 1 + 2)$ which is $(1,0)$.

(e) The game ends when all points are covered. The winner is the player with the most points covered. As you play the game you will see that it involves several interesting strategies.

7.10 Sets of Lattice Points and Mappings of Z into Z

You are familiar with many types of mappings from Chapter 3. An important use of lattice points is the representation of mappings of Z into Z.

The diagram below displays some of the assignments made by $x \longrightarrow 2x$ where $\underline{x}$ is a member of Z.

$$\begin{tikzcd}
\text{Domain} & \dots & -3 \arrow[d] & -2 \arrow[d] & -1 \arrow[d] & 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \dots \\
\text{Range} & \dots & -6 & -4 & -2 & 0 & 2 & 4 & 6 & \dots
\end{tikzcd}$$

The pairs associated by the mapping can also be displayed as a subset of Z X Z.

[... , (-3,-6), (-2,-4), (-1,-2), (0,0), (1,2), (2,4), (3,6), ...]

This subset can be graphed:

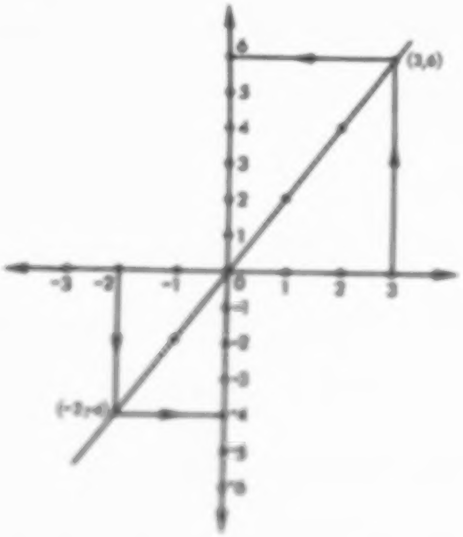


Figure 7.13

In this particular mapping we see that points of interest are those (x,y) where y = 2x. The arrow from 3 on the x-axis to the point (3,6) and the arrow from the point (3,6) to the point 6 on the y-axis illustrate a geometric method of using the graph to find the integer on the y-axis assigned to a particular integer selected from the x-axis.

Select some other integers from the domain of the mapping illustrated in figure 7.12 and for each trace the path from the point on the x-axis to the point in the graph, and then over to the corresponding member of the range on the y-axis.

Which axis contains the graph of the domain of a mapping?

Which axis contains the graph of the range of a mapping?

The condition $y = \frac{12}{x}$ gives rise to the mapping $x \rightarrow \frac{12}{x}$, if we restrict the domain of the mapping to the set T of integers that divide 12. Thus

$$T = \{-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12\}$$

To graph the mapping, we proceed as follows: Take an element of T, say -6. Compute $\frac{12}{x}$, in this case $\frac{12}{-6} = -2$. Under the mapping $x \rightarrow \frac{12}{x}$, x is assigned $\frac{12}{x}$. Hence -6 is assigned -2, and the ordered pair (-6, -2) is in the graph of the mapping. (We may think of this as follows: $y = \frac{12}{x}$. Take $x = -6$. Then $y = \frac{12}{-6} = -2$, and the pair $(x,y) = (-6,-2)$ is determined.) If we take the element 4 from the domain of the mapping, then $y = \frac{12}{4} = 3$, and (4,3) is a point in the graph.

In this manner we can find other pairs and record them in a table.

Domain	Range
-12	
-6	-2
-4	
-3	
-2	
-1	
1	
2	
3	
4	3
6	
12	

Copy and complete the table above. Draw axes on a sheet of graph paper and circle the points obtained from the table.

7.11 Exercises

1. Make a table like that above for each of the following open sentences:

(a) $y = x^2$ (d) $y = 2x - 1$

(b) $y = 2x + 1$ (e) If x is even, $y = 9$; and if x is odd, $y = 1$.

(c) $y = x^2 + 1$

2. Use the tables that you constructed in Exercise 1 to circle the points in the graph of each condition. Use graph paper and make a separate pair of axes for each graph.

*7.12 Lattice Points in Space

If Z represents the set of integers, and Z X Z represents the set of all ordered pairs of integers, what do you think Z X Z X Z represents?

You have seen that Z may be associated with a set of points on a line and that Z X Z may be associated with a set of points in a plane. The set of all ordered triples of integers may be associated with a set of points in (3-dimensional) space.

Suppose that you wish to meet a friend in an office building on the corner of some avenue and street. You not only need to know the number of the street and the number of the avenue, but also the number of the floor in the office building.

The longitude and latitude of an airplane at any instant is not sufficient to determine its position. You also need to know its altitude.

In each of these examples, it is necessary to have a triple of numbers to locate an object in space. In a corresponding way, we associate each point in a three-dimensional set of lattice points with an ordered triple of integers. In this case we have three axes instead of two and each point has three coordinates.

Figure 7.14 illustrates the assignment of coordinates to certain points in space. Study the diagram and see if you can discover how each triple (x,y,z) was assigned.

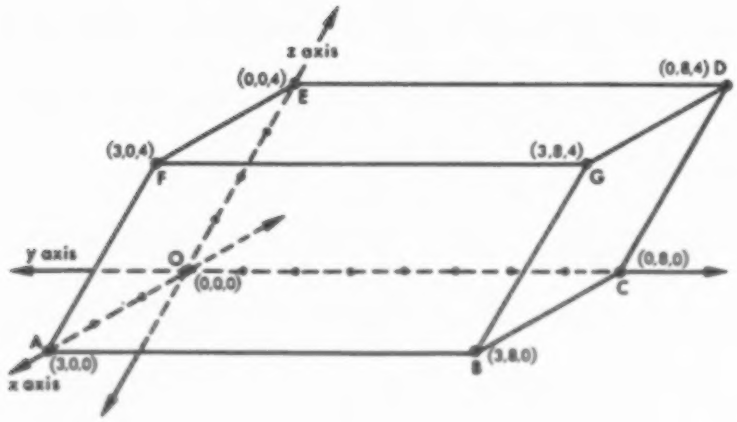


Figure 7.14

The geometric figure with vertices A,B,G,F is a parallelogram because line AB is parallel to line FG, and line FA is parallel to line GB. The geometric figure with vertices OABCDEFG has six faces each of which is a parallelogram. It is called a parallelepiped.

7.13 Exercises

1. (a) Name the six faces of the parallelepiped using the letters that name the vertices.
   (b) How many of the parallelograms have O as a vertex?
   (c) Try to draw the parallelepiped that has O as a vertex for three of its faces and has the point (2,3,4) as the other end of the diagonal from O. List the coordinates of all 8 vertices.
2. (a) Using three pieces of cardboard, try to construct a model of three planes so that any pair of planes has

a line in common, but all three have only one point in common.

7.14 Translations and Z X Z

In Chapter 4 you learned about translations as a special kind of mapping. You also learned that the set of translations in a line, as represented by directed numbers, with the operation "following" has the properties of a commutative group.

In this chapter we will be chiefly interested in translations of a set of lattice points into itself in terms of coordinates.

We will designate the image of point P in a mapping by "P′" (read: P-prime). If the coordinates of P are (x,y), then the coordinates of P′ are (x',y').

Translations "move" every point in the lattice the same distance in the same direction.

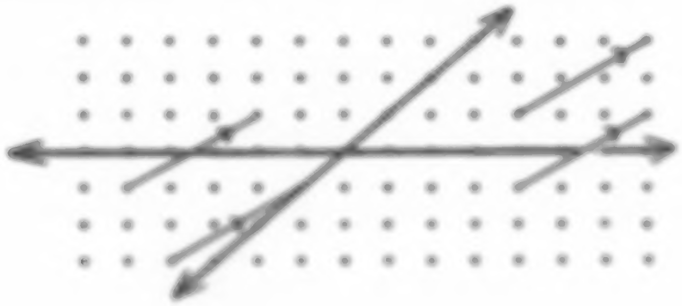


Figure 7.15

The diagram in Figure 7.15 shows the effect of a certain translation on four points:

$(-4,-1) \longrightarrow (-3,1)$

$(-1,-3) \longrightarrow (0,-1)$

$$(5,-1) \longrightarrow (6,1)$$

$$(3,1) \longrightarrow (4,3)$$

Questions: In each case, by how much did the first coordinate increase?

By how much did the second coordinate increase?

What is the image of the following points in the same translation?

(a) (2,3) (b) (6,-2) (c) (-1,2) (d) (0,0)

The above translation may be defined by:

$$(x,y) \longrightarrow (x + 1,\ y + 2) \text{ or by } T_{1,2}$$

$T_{1,2}$ indicates that the translation adds 1 to the first coordinate of each point and 2 to the second coordinate.

Any translation of Z X Z may be designated

$$(x,y) \longrightarrow (x + a,\ y + b) \text{ or } T_{a,b}$$

where a and b are integers.

What would be the effect of the translation $T_{0,0}$? Since $T_{0,0}$ or $(x,y) \longrightarrow (x + 0,\ y + 0)$ maps every point onto itself, it is called the identity translation.

You are familiar with the composition of mappings. In connection with translations of a set of lattice points the composition of $T_{a,b}$ with $T_{c,d}$ can be expressed as:

$$T_{a,b} \circ T_{c,d} = T_{c+a,d+b}$$

The symbol "o" in the definition above can be read "with" or "following" since the translation on the right of the "o" translates first. The effect of the above composition of translations on a point (x,y) is:

$$(x,y) \longrightarrow (x + c + a,\ y + d + b)$$

If you placed a disk on the lattice point in the coordinate system in Figure 7.16 the composition $T_{2,3} \circ T_{-4,-1}$ would tell you to first move the disk 4 points to the left and one down, and follow this by 2 to the right and 3 up. Since $T_{2,3} \circ T_{-4,-1} = T_{-2,2}$ this should be the same as moving 2 to the left and 2 up. Figure 7.16 illustrates this by showing the effect on (0,0).

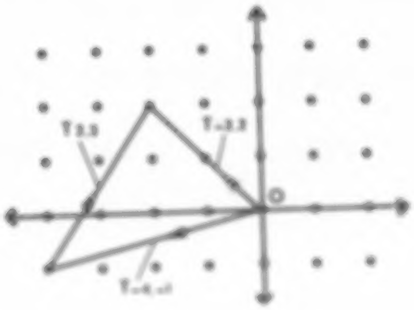


Figure 7.16

7.15 Exercises

1. Find the compositions of the following pairs of translations:

   (a) $T_{[illegible]} \circ T_{[illegible]}$ (b) $T_{[illegible]} \circ T_{[illegible]}$

   (c) $T_{a,b} \circ T_{-a,-b}$

   If the composition of two translations is the identity translation, each is called the inverse of the other.

2. Use the commutative property for addition of integers to show that $T_{a,b} \circ T_{c,d} = T_{c,d} \circ T_{a,b}$.

3. What property does Exercise 2 demonstrate for composition of translations?

4. Use a property of integers to show the following:
$T_{a,b} \circ (T_{c,d} \circ T_{e,f}) = (T_{a,b} \circ T_{c,d}) \circ T_{e,f}$.

5. What property of composition of translations is demonstrated in Exercise 4?

6. Draw the parallelogram with the following vertices on graph paper:
(-3,-1), (0,3), (7,3), (4,-1)

7. Verify with a ruler that the midpoint of each pair of opposite vertices in Exercise 6 is (2,1).

8. Find the images of the points in Exercise 6 under the translation $T_{-2,-1}$: that is map each (x,y) onto (x - 2, y - 1).

9. Verify that the image points you found in Exercise 8 form the vertices of another parallelogram.

10. Verify with a ruler that the midpoint of each pair of opposite vertices in Exercise 9 is the image of the point (2,1) under the translation $T_{-2,-1}$.

7.16 Dilation and Z X Z

Figure 7.17 shows graphically what happens to a set of points under dilation.

A dilation of Z X Z is a mapping designated by

$$(x,y) \longrightarrow (ax,ay) \text{ or } D_a,$$

for any non-zero integer a.

In the dilation in Figure 7.17 a = 2. The mapping is $(x,y) \longrightarrow (2x,2y)$, or D_2. An equivalent way to say this is that distances between pairs of points in the image are twice as great as the distances between the corresponding pairs of points in the first picture. If the dilation had been D_{-2}, the image would have been the same size but would have been upside down below the x-axis and to the left of the y-axis, with his nose still against the y-axis but 6 units below the origin.

Exercise: Dilate the original picture by a factor of -2. Then the mapping is $(x,y) \longrightarrow (-2x, -2y)$. You will see him increase in size and stand on his head!

In any dilation both coordinates of each point are multiplied by the same number. We will refer to this number as "a" in the following questions:

(a) What happens to points in a dilation if a = 1?

(b) What happens to points in a dilation if a = -1?

(c) If we should allow a to be zero, onto which point would each point map?

(d) What happens to each point in a set of points if a = 3? If a = -3?

(e) If a picture is to the left of the y-axis and above the x-axis in Figure 7.17, where will the image be under D_2? Under D_{-2}?

(f) Where will any point on the x-axis be mapped by a dilation $(x,y) \longrightarrow (ax,ay)$? Where will a point on the y-axis be mapped?

6

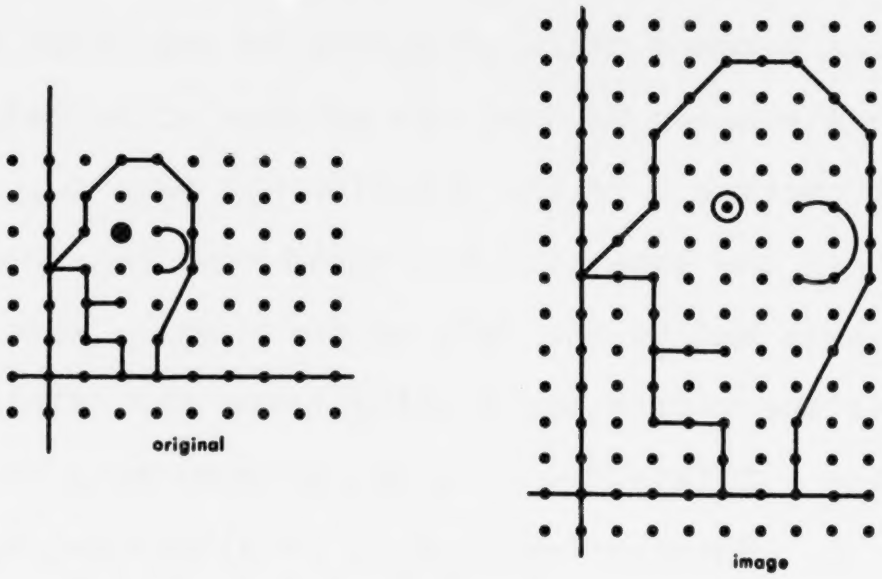


Figure 7.17

7.17 Exercises

1. Use the dilation D_3 to graph the following points and their images:

   (-3,-1), (0,3), (7,3), (4,-1)

2. Answer the following questions about the figure in Exercise 1 and its image:
   (a) What kind of geometric figure do the four given points outline?
   (b) Do the image points outline a figure the same size as the original? The same shape?

3. The composition of dilations may be represented as $D_b \circ D_a = D_{ab}$ where D_a dilates first.
   (a) Which dilation maps every point onto itself?

(c) Which two dilations are the only ones that have inverses in Z X Z?

*7.18 Some Additional Mappings and Z X Z

By now you should have some skill in finding images if you are given the coordinates of a point and a rule for finding the image. For each of the mappings below, find the images of the following points which outline a parallelogram and the midpoint of its opposite vertices. Then answer questions (a) - (g).

Points: (-3,-1), (0,3), (7,3), (4,-1)

Midpoint of opposite vertices: (2,1)

(a) First use graph paper to graph the figure and its image.

(b) Does the image outline another parallelogram?

(c) Is the image of the midpoint the midpoint of the images of the opposite vertices?

(d) Do the image points outline a figure the same size as the original? the same shape?

(e) If the vertices of the parallelogram are named ABCD clockwise in that order, are their respective images A', B', C', D' also in clockwise order?

(f) For each mapping compose the mapping with itself.

(g) Compose the following mappings: (1) following (2), (3) following (5), (4) following (6), (5) following (6), (6) following (5).

Mappings:

(1) $(x,y) \longrightarrow (x,-y)$

(2) $(x,y) \longrightarrow (-x,y)$

(3) $(x,y) \longrightarrow (y,x)$

(4) $(x,y) \longrightarrow (y,-x)$

(5) $(x,y) \longrightarrow (x + 3, -y)$

(6) $(x,y) \longrightarrow (x + 2y, y)$

7.19 Summary

1. The assignment of ordered pairs of integers to lattice points in a plane involves
   (a) assignment of integers to equally spaced points on each of two intersecting lines called axes;
   (b) assignment of pairs of integers, one from each axis, to lattice points in the plane of axes.
2. The set of all ordered pairs of integers is named Z X Z, and the two integers assigned to a point are called coordinates of the point.
3. Conditions on coordinates of a point, such as "the sum of the coordinates is 3," are expressed by open sentences, such as "$x + y = 3$." The set of ordered pairs, each of which satisfies the condition, is called the solution set of the condition (or the open sentence). The set of lattice points that have these pairs for coordinates is the graph of the condition.
4. Compound conditions may be expressed by connecting two open sentences with "and." The connective "or" can also

be used. A pair of integers satisfies an "and" condition if it satisfies both connected conditions. It satisfies an "or" condition, if it satisfies either.

5. The absolute value of an integer is defined by

   $|x| = x$, if $x \geq 0$,

   $|x| = -x$, if $x < 0$.

6. The idea of a coordinate system in a plane may be extended to space by assigning number triples to points.

7. Translations of Z X Z are expressed by

   $(x,y) \longrightarrow (x + a,\ y + b)$.

8. Dilations of Z X Z are expressed by

   $(x,y) \longrightarrow (ax,ay)$.

7.20 Review Exercises

1. List five ordered pairs of integers that satisfy the condition:

   (a) $x + 2y = 5$ (b) $x = 27$ (c) $y = |x| - 2$

   (d) $|x| + |y| = 3$ (e) $xy = 24$

2. Translate the following conditions into open sentences:

   (a) Two times the first coordinate minus three times the second coordinate is equal to seven.

   (b) The first coordinate is three less than two times the absolute value of the second coordinate.

   (c) The first coordinate is greater than zero and the second coordinate is less than two.

3. Translate the following open sentences into words:

   (a) $y = x^2 - 2$ (b) $|x + y| = 5$ (c) $y > 2$ or $x < 3$.

4. Tabulate the solution set of the following:

   (a) $x + y = 5$ and $x - y = 3$. (b) $y = x^2$ and $x = -1$.

5. Graph the following:

   (a) $y = 2x - 1$ (b) $y = -3x$ (c) $x > 0$ and $y = 0$.

6. Which "region" or "regions" contain the points whose coordinates satisfy the following:

   (a) $x = 2$ and $y > 0$. (b) (x,y) is not on either axis.

   (c) $y < -5$ and $x < -6$. (d) $x = -10$ and $y = 23$.

7. Draw a pair of axes on a sheet of graph paper and circle the following points:

   (6,11), (6,1), (11,6), (1,6), (9,10), (3,10), (3,2), (9,2), (10,9), (10,3), (2,3), (2,9)

8. Find the image of each point in Exercise 7 for the following mappings, and circle the image points:

   (a) $(x,y) \longrightarrow (x,-y)$ (c) $(x,y) \longrightarrow (-2x,-2y)$

   (b) $(x,y) \longrightarrow (-x,y)$ (d) $(x,y) \longrightarrow (y,x)$

9. On a sheet of graph paper, mark the following points, and draw the triangle they outline:

   (0,0), (0,5), (2,0).

   On the same sheet of graph paper mark the images of the three points under the following mappings, and in each case draw the triangle the three image points outline.

   (a) $(x,y) \longrightarrow (2x,2y)$ (b) $(x,y) \longrightarrow (-2x,2y)$

   (c) $(x,y) \longrightarrow (-2x,-2y)$ (d) $(x,y) \longrightarrow (2x,-2y)$

10. On a sheet of graph paper, mark the following four points, and draw the quadrilateral they outline:

(0,0), (0,3), (4,3), (4,0)

On the same sheet of graph paper mark the images of the four points under the following mappings, and in each case draw the quadrilateral the four image points outline:

(a) $(x,y) \longrightarrow (x + 3, y + 4)$ (b) $(x,y) \longrightarrow (x + 2y, y)$

(c) $(x,y) \longrightarrow (x + 5, -y)$ (d) $(x,y) \longrightarrow (x,0)$

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